Regional variance in target number: analysis and application for multi-Bernoulli point processes

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Abstract. In the context of multi-target tracking application, the concept of variance in the number of targets estimated in specified regions of the surveillance scene has been recently introduced for multi-object filters. This article has two main objectives. First, the regional variance is derived for a multi-object representation commonly used in the tracking literature, known as the multi-Bernoulli point process, in which the multi-target state is described with a set of hypothesised tracks with associated existence probabilities. This model is exploited in multi-target applications where it can be assumed that targets evolve independently of each other and generate sensor observations that are uncorrelated with other targets. An illustration of the concept of regional statistics (mean and variance) in target number, and how to interpret them in the broader context of multi-object filtering, it then provided. Possible applications include performance assessment and sensor control for multi-target tracking.

1. Introduction

In the last decade, the field of sensor fusion has witnessed a paradigm shift in the way that methods for tracking multiple targets in defence-related surveillance applications are developed. Heuristic approaches for estimating multiple dynamical objects have been developed since the 1970s, yet these methods suffer from systematic failure due to the heuristics introduced for track management. A radically different approach based on Random Finite Sets considers the problem in a unified way that enables operators to estimate the correct number of targets in challenging environments where there may be many false alarms and the targets are not always observed. This led to principled low computational cost approximate solutions that could be deployed on real-time systems known as multi-object filters [1–6].

In the multi-object filtering framework developed during the last decade, the multi-target configuration (i.e. the multiple targets evolving in the surveillance scene) is described by a random finite set or (simple) point process [7]. One of the most popular approaches for multi-object filtering relies on propagating the first moment of this point process, known as the Probability Hypothesis Density (PHD), from which the expected number of targets in any region of the surveillance scene can be extracted [1, 2]. Recent work demonstrates that, using recent developments in multi-object filtering [8–10], the second moment of this point process can be computed as well in order to provide uncertainty on the expected number of targets in any region [11]. The concept of regional variance is introduced and illustrated in [11] for the PHD filters, which are based on multi-target representations through Poisson and i.i.d. point processes.

Bernoulli and multi-Bernoulli point processes [7] are another example of multi-target representation widely used in the tracking literature. The Bernoulli process models a single target with a probability distribution describing its state (position, velocity, etc.), and an existence probability depicting its credibility. A similar representation was pioneered by Musicki's work on the Integrated Probabilistic Data Association (IPDA) filter [12], and later integrated to the multi-object filtering framework as a Bernoulli point process for joint detection and tracking filters [7, 13–17]. The extension to multiple objects, known as the multi-Bernoulli process, has been developed for multi-object filtering [7, 18, 19]. It has been applied for visual tracking in images [20–25], or audio-video fusion [26], and inspired numerous implementations (see [27–45] for recent examples). In a more general context, (multi-)Bernoulli filters have also been developed for scenarios with unknown clutter intensity [46], sensor control problems [15, 47], tracking in sensor networks [48–50], superpositional sensors [51, 52], distributed data fusion [53–55], and tracking with road constraints [56]. Recent developments introduced track labelling in multi-Bernoulli processes and enable closed-form solutions to multi-target tracking problems assuming target independency [57–62].

The increasing level of interest for the modelling of multiple targets through Bernoulli processes motivates a study for its statistical and analytical properties. One of the main objective of this article is to derive the regional variance for a generic (multi-)Bernoulli process, a novel statistical tool that has been recently introduced and exploited for the study of the PHD filters [11]. In complement, this article discusses and illustrates the concept of regional variance as a generic assessment tool for multiple-target tracking algorithms derived within the multi-object filtering framework.

The structure of the article is as follows: Section 2. formally defines a point process used for the stochastic description of population of targets, and Section 3. describes the evolution of a target point process in the Bayesian filtering paradigm. Section 4. describes the method to produce the regional mean and variance in target number for a generic point process, and Section 5. introduces the functional representation of point processes. Section 6. describes the construction of the regional statistics for the Bernoulli and multi-Bernoulli processes, and discusses the given results. Section 7. proposes an interpretation of the regional statistics for the assessment of multi-object filters, and the article concludes in Section 8..

2. Point processes

In this section, we introduce the notations for point processes used in the article and highlight their relation to Finite Set Statistics [7]. Assume that a surveillance scene contains objects of interest or *targets*, which are individually described by a state x in some target space $\mathscr{X} \subset \mathbb{R}^{d_x}$ (position, velocity, etc.). Because 1) new targets may enter the scene or be spawned from existing ones, and 2) some targets may leave the scene or disappear by other means, the target number and the target states may vary across time.

In the context of multi-target detection and tracking problems, an operator is interested in estimating the target number *and* the target states. Since both are unknown a priori, the target population is described at any instant by a point process Φ whose number of elements and element states are random. A realisation of a point process Φ is a set of points $\varphi = \{x_1, \dots, x_N\}$ that describes a particular state of the target population, i.e. a multi-target configuration.

A point process Φ on $\mathscr{X}$ is defined formally through a measurable mapping:

$$\Phi : (\Omega, \mathscr{F}, \mathbb{P}) \to (E_{\mathscr{X}}, \mathbf{B}_{E_{\mathscr{X}}}) \tag{1}$$

from some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ to the measurable space $(E_{\mathscr{X}}, \mathbf{B}_{E_{\mathscr{X}}})$, where $E_{\mathscr{X}}$ is the point process state space, i.e., the space of all the finite sets of points in $\mathscr{X}$, and $\mathbf{B}_{E_{\mathscr{X}}}$ is the Borel σ-algebra on $E_{\mathscr{X}}$ [63]. As for usual random variables, a point process Φ is more conveniently described by the probability distribution on $(E_{\mathscr{X}}, \mathbf{B}_{E_{\mathscr{X}}})$ generated by $\mathbb{P}$ and denoted by P_Φ.

3. Multi-target Bayesian filtering

In multi-target detection and tracking problems, the operator updates their knowledge of the multi-target configuration through successive sequences of observations $z^k_{1:m_k}$ produced at discrete time step $k > 0$ by some sensor system observing the scene. The target process $\Phi_{k|k}$ is a point process providing a stochastic description of the posterior distribution of the targets in the state space at time k, based on the measurement history up to time k.

The usual single-object Bayesian paradigm is generalisable to the multi-object framework [7]. The law of the filtered state $P_{\Phi_{k|k}}$ is updated through sequences of *prediction steps* and *data update steps*. The full multi-target Bayes' filter is then found to be [7]:

$$P_{\Phi_{k|k-1}}(d\xi) = \int T_{k|k-1}(d\xi|\varphi) P_{\Phi_{k-1|k-1}}(d\varphi), \tag{2}$$

$$P_{\Phi_{k|k}}(d\xi|z^k_{1:m}) = \frac{L_k(z^k_{1:m}|\xi) P_{\Phi_{k|k-1}}(d\xi)}{\int L_k(z^k_{1:m}|\varphi) P_{\Phi_{k|k-1}}(d\varphi)}, \tag{3}$$

where $T_{k|k-1}$ is the Markov transition kernel between time steps $k-1$ and k, and L_k is the multi-measurement/multi-target likelihood at time step k.

4. Regional statistics: mean and variance in target number

While a point process is fully characterized by its probability distribution P_Φ, i.e. its realisations in the process state space $E_{\mathscr{X}}$, an alternative description is available through the point patterns it produces in the target state space $\mathscr{X}$. For any Borel set $B \in \mathbf{B}_{\mathscr{X}}$, where $\mathbf{B}_{\mathscr{X}}$ is the Borel σ-algebra on $\mathscr{X}$, the counting measure

$$N_\Phi(B) = \sum_{x \in \Phi} 1_B(x) \tag{4}$$

counts the number of targets falling inside B according to the point process Φ [63]. Using the well-defined statistical moments of the integer-valued random variables $N_\Phi(B)$ for any $B \in \mathbf{B}_{\mathscr{X}}$, one can define the *moment measures* of the point process Φ.

For any regions $B, B' \in \mathbf{B}_{\mathscr{X}}$, the first and second moment measures $\mu^{(1)}_\Phi$, $\mu^{(2)}_\Phi$ are defined by

$$\mu^{(1)}_\Phi(B) = \mathbb{E}[N_\Phi(B)] \tag{5a}$$

$$= \int \left(\sum_{x \in \varphi} 1_B(x) \right) P_\Phi(d\varphi) \tag{5b}$$

$$= \sum_{n \geqslant 0} \int \left(\sum_{1 \leqslant i \leqslant n} 1_B(x_i) \right) P_\Phi(dx_{1:n}), \tag{5c}$$

and

$$\mu^{(2)}_\Phi(B, B') = \mathbb{E}\left[N_\Phi(B) N_\Phi(B')\right] \tag{6a}$$

$$= \int \left(\sum_{x_i, x_j \in \varphi} 1_B(x_i) 1_{B'}(x_j) \right) P_\Phi(d\varphi) \tag{6b}$$

$$= \sum_{n \geqslant 0} \int \left(\sum_{1 \leqslant i,j \leqslant n} 1_B(x_i) 1_{B'}(x_j) \right) P_\Phi(dx_{1:n}), \tag{6c}$$

where $x_{1:n} = \{x_1, \dots, x_n\}$, $\mathbb{E}$ is the expectation operator, and 1_B is the indicator set function defined by

$$\forall x \in \mathscr{X}, 1_B(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases} \tag{7}$$

The first moment measure $\mu^{(1)}_\Phi(B)$ provides the expected number of targets or *mean target number* inside B, while $\mu^{(2)}_\Phi(B, B')$ denotes the joint expectation of the target number inside B and B'. Similarly to usual random variables, the *variance* var_Φ of the point process Φ [63] in any region $B \in \mathbf{B}_{\mathscr{X}}$ is defined by

$$\mathrm{var}_\Phi(B) = \mu^{(2)}_\Phi(B, B) - \left[\mu^{(1)}_\Phi(B)\right]^2. \tag{8}$$

Take notice that the variance is a function, but not a *measure*, on the Borel σ-algebra $\mathbf{B}_{\mathscr{X}}$. It does not necessarily admit a density, in general, even if $\mu_\Phi^{(2)}$ and $\mu_\Phi^{(1)}$ do. For this reason, this article and the previous work on the regional statistics for PHD filters [11] adopt a measure-theoretical approach to the study of point processes covered in [10]. In particular, we study a point process Φ through its first moment *measure* $\mu_\Phi^{(1)}$ rather than its first moment *density* or Probability Hypothesis Density, even though the latter choice is more common in recent studies [7,61,62].

The *regional statistics* $(\mu_\Phi^{(1)}(B), \mathrm{var}_\Phi(B))$ provide an *approximate* description of $N_\Phi(B)$, i.e. the number of target in B according to the point process Φ:

- $\mu_\Phi^{(1)}(B)$ is the mean (or expected) target number within B;
- $\mathrm{var}_\Phi(B)$ quantifies the dispersion of the target number within B around its mean value.

More details on the practical exploitation of the regional statistics are provided in section 7..

5. Functional representation

Rather than relying directly on the probability distribution P_Φ to represent the point process Φ, it is possible to introduce "characteristic" functionals with convenient properties. An example of such a characteristic functional can be found to be the Laplace functional $\mathscr{L}_\Phi$ [64], defined by the expectation

$$\mathscr{L}_\Phi[f] = \mathbb{E}\left[\prod_{x\in\Phi} e^{-f(x)}\right] \tag{9a}$$

$$= \int \exp\left(-\sum_{x\in\varphi} f(x)\right) P_\Phi(d\varphi). \tag{9b}$$

The expression of different statistical quantities for a given point process Φ can be found by functional differentiation of $\mathscr{L}_\Phi$. For this purpose, we use an appropriate restriction of the Gâteaux differential, named the chain differential [65], for which it exists a composition rule. The chain differential $\delta F(h;\eta)$ of a functional F, (evaluated) at function h in the direction (or increment) η, is defined as

$$\delta F(h;\eta) = \lim_{n\to\infty} \frac{F(h+\varepsilon_n\eta_n) - F(h)}{\varepsilon_n}, \tag{10}$$

where $\{\eta_n\}_{n\geqslant 0}$ is a sequence of functions η_n converging (pointwise) to η, $\{\varepsilon_n\}_{n\geqslant 0}$ is a sequence of positive real numbers converging to zero, if the limit exists and is identical for any admissible sequences $\{\eta_n\}_{n\geqslant 0}$ and $\{\varepsilon_n\}_{n\geqslant 0}$.

Using the chain differential we find that the first and second moment measures in any regions $B, B' \in \mathbf{B}_\mathscr{X}$ can be recovered from the Laplace functional $\mathscr{L}_\Phi$ when differentiating once or twice in the point $f = 0$ as follows

$$\mu_\Phi^{(1)}(B) = -\,\delta(\mathscr{L}_\Phi[f];1_B)\big|_{f=0}, \tag{11}$$

$$\mu_\Phi^{(2)}(B,B') = \delta^2(\mathscr{L}_\Phi[f];1_B,1_{B'})\big|_{f=0}. \tag{12}$$

It is also possible to simplify the Laplace functional by considering the test function $f = -\log h$. The resulting functional $\mathscr{G}$ with argument h is called the Probability Generating Functional (PGFl) and is found to be

$$\mathscr{G}_\Phi[h] = \mathscr{L}_\Phi[-\log h], \tag{13}$$

where h is a test function, i.e., a real-valued function belonging to the space of bounded measurable functions on $\mathscr{X}$, such that $0 \leqslant h(x) \leqslant 1$ and $1-h$ vanishes outside some bounded region of $\mathscr{X}$ [64].

The PGFl $\mathscr{G}_\Phi[h]$ also characterises the point process Φ and can be expressed as

$$\mathscr{G}_\Phi[h] = \mathbb{E}\left[\prod_{x\in\Phi} h(x)\right] \tag{14a}$$

$$= \int \left(\prod_{x\in\varphi} h(x)\right) P_\Phi(d\varphi). \tag{14b}$$

The first moment measure $\mu_\Phi^{(1)}$ of the point process Φ can now be recovered via the differentiation of the PGFl $\mathscr{G}_\Phi$ as follows:

$$\mu_\Phi^{(1)}(B) = \delta(\mathscr{G}_\Phi[h];1_B)\big|_{h=1}, \tag{15}$$

for any subset B in $\mathbf{B}_\mathscr{X}$. However, the variance cannot be directly recovered from the PGFl, and still requires the generality of the Laplace functional to be calculated.

6. Regional statistics for Bernoulli and multi-Bernoulli point processes

6.1 Bernoulli point process

A Bernoulli process [7, 14] is an elementary point process Φ whose number of elements is *at most one*. A realisation φ is either:

- The empty set $\emptyset$, with probability $1 - P_\mathrm{e}$,
- A single object within some infinitesimal neighbourhood $dx \in \mathbf{B}_\mathscr{X}$, with probability $P_\mathrm{e} p_\mathrm{s}(dx)$.

Consequently, the probability of existence $0 \leqslant P_\mathrm{e} \leqslant 1$ and the spatial distribution p_s on $(\mathscr{X}, \mathbf{B}_\mathscr{X})$ fully characterise the process Φ. In the context of target tracking, a Bernoulli provides a natural description for a potential target or *track*. The probability of existence P_e denotes the confidence of the operator in the track, and the spatial distribution p_s the information acquired by the operator about the state of the potential target.

Following (15) and (12), one must first determine the PGFl $\mathscr{G}_\Phi$ and the Laplace functional $\mathscr{L}_\Phi$ of the process Φ to produce the moment measures. Since the size of a realisation φ is at most one, the expressions of the PGFl (14b) and the Laplace functional (9b) simplify and yield

$$\mathscr{G}_\Phi[h] = 1 - P_\mathrm{e} + P_\mathrm{e}\int h(x) p_\mathrm{s}(dx) \tag{16}$$

$$\mathscr{L}_\Phi[h] = 1 - P_\mathrm{e} + P_\mathrm{e}\int e^{-f(x)} p_\mathrm{s}(dx) \tag{17}$$

Using (15), we first derive the PGFl (16) once to produce the first moment measure in any region $B \in \mathbf{B}_{\mathscr{X}}$:

$$\mu_\Phi^{(1)}(B) = \delta(\mathscr{G}_\Phi[h]; 1_B)|_{h=1} \tag{18a}$$
$$= P_\mathrm{e} \int \delta(h(x); 1_B)|_{h=1} p_\mathrm{s}(dx) \tag{18b}$$
$$= P_\mathrm{e} \int 1_B(x) p_\mathrm{s}(dx), \tag{18c}$$

where we draw the result $\delta(h(x); 1_B)|_{h=1} = 1_B(x)$ from Corollary 1 in [8]. Using (12), we then derive the Laplace functional (17) twice to produce the second moment measure in any regions $B, B' \in \mathbf{B}_{\mathscr{X}}$:

$$\mu_\Phi^{(2)}(B, B') = \delta^2(\mathscr{L}_\Phi[f]; 1_B, 1_{B'})|_{f=0} \tag{19a}$$
$$= P_\mathrm{e} \int \delta^2(e^{-f(x)}; 1_B, 1_{B'})|_{f=0} p_\mathrm{s}(dx) \tag{19b}$$
$$= P_\mathrm{e} \int 1_B(x) 1_{B'}(x) p_\mathrm{s}(dx) \tag{19c}$$
$$= P_\mathrm{e} \int 1_{B \cap B'}(x) p_\mathrm{s}(dx), \tag{19d}$$

where we draw the result $\delta^2(e^{-f(x)}; 1_B, 1_{B'})|_{f=0} = 1_B(x)1_{B'}(x)$ from Appendix B-E in [11].

Note that the second moment measure (19d) evaluated at $B = B'$ equals the first moment measure (18c) evaluated in B. This is not a general result to point processes, but a specific result of the Bernoulli process. Finally, the definition of the variance (8) yields

$$\mathrm{var}_\Phi(B) = \mu_\Phi^{(2)}(B, B) - \left[\mu_\Phi^{(1)}(B)\right]^2 \tag{20a}$$
$$= \mu_\Phi^{(1)}(B)\left(1 - \mu_\Phi^{(1)}(B)\right). \tag{20b}$$

6.2 Multi-Bernoulli point process

A multi-Bernoulli point process Φ [7, 66] is constructed through the superposition of independent Bernoulli point processes $\Phi_1, \ldots, \Phi_n$, i.e. $\Phi = \Phi_1 \cup \ldots \cup \Phi_n$ or, in terms of the counting measures,

$$N_\Phi(B) = \sum_{i=1}^{n} N_{\Phi_i}(B). \tag{21}$$

In the context of multi-target tracking, each Bernoulli process Φ_i represents a hypothesised track, with associated probability of existence $P_{\mathrm{e},i}$ and estimated state distributed according to $p_{\mathrm{s},i}$. Provided that the assumption on target independency holds – i.e. the targets evolve and generate observations independently of each other – a multi-Bernoulli process provides an appropriate description of the multi-target configuration. Besides, the PGFl of the multi-Bernoulli is given by the product of the PGFls of the Bernoulli processes [7]; using (13) the same relation holds for the Laplace functionals and we get

$$\mathscr{G}_\Phi[h] = \prod_{i=1}^{n} \mathscr{G}_{\Phi_i}[h] \tag{22}$$

$$\mathscr{L}_\Phi[f] = \prod_{i=1}^{n} \mathscr{L}_{\Phi_i}[f] \tag{23}$$

Using (15), we first derive the PGFl (22) once to produce the first moment measure in any region $B \in \mathbf{B}_{\mathscr{X}}$:

$$\mu_\Phi^{(1)}(B) = \delta\Big(\prod_{i=1}^{n} \mathscr{G}_{\Phi_i}[h]; 1_B\Big)\Big|_{h=1} \tag{24a}$$
$$= \sum_{i=1}^{n} \delta(\mathscr{G}_{\Phi_i}[h]; 1_B)|_{h=1} \prod_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \mathscr{G}_{\Phi_j}[h]|_{h=1}, \tag{24b}$$

where $\mathscr{G}_{\Phi_j}[1] = 1$ using (16), and thus

$$\mu_\Phi^{(1)}(B) = \sum_{i=1}^{n} \delta(\mathscr{G}_{\Phi_i}[h]; 1_B)|_{h=1} \tag{24c}$$
$$= \sum_{i=1}^{n} \mu_{\Phi_i}^{(1)}(B). \tag{24d}$$

We then move to the second moment measure. Using (12), we derive the Laplace functional (17) twice to produce the second moment measure in any regions $B, B' \in \mathbf{B}_{\mathscr{X}}$:

$$\mu_\Phi^{(2)}(B, B') = \delta^2\Big(\prod_{i=1}^{n} \mathscr{L}_{\Phi_i}[f]; 1_B, 1_{B'}\Big)\Big|_{f=0} \tag{25a}$$
$$= \sum_{i=1}^{n} \delta^2(\mathscr{L}_{\Phi_i}[f]; 1_B, 1_{B'})|_{f=0} \prod_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} \mathscr{L}_{\Phi_j}[f]|_{f=0} \tag{25b}$$
$$+ \prod_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \delta(\mathscr{L}_{\Phi_i}[f]; 1_B)|_{f=0} \delta(\mathscr{L}_{\Phi_j}[f]; 1_{B'})|_{f=0} \prod_{\substack{1 \leqslant k \leqslant n \\ k \neq i,j}} \mathscr{L}_{\Phi_k}[f]|_{f=0} \tag{25c}$$

where $\mathscr{L}_{\Phi_j}[0] = 1$ using (17), and thus

$$\mu_\Phi^{(2)}(B, B') = \sum_{i=1}^{n} \delta^2(\mathscr{L}_{\Phi_i}[f]; 1_B, 1_{B'})|_{f=0} + \prod_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \delta(\mathscr{L}_{\Phi_i}[f]; 1_B)|_{f=0} \delta(\mathscr{L}_{\Phi_j}[f]; 1_{B'})|_{f=0} \tag{25d}$$
$$= \sum_{i=1}^{n} \mu_{\Phi_i}^{(2)}(B, B') + \prod_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \mu_{\Phi_i}^{(1)}(B) \mu_{\Phi_j}^{(1)}(B'). \tag{25e}$$

The definition of the variance (8) then yields

$$\mathrm{var}_\Phi(B) = \mu_\Phi^{(2)}(B, B) - \left[\mu_\Phi^{(1)}(B)\right]^2 \tag{26a}$$
$$= \sum_{i=1}^{n} \mu_{\Phi_i}^{(2)}(B, B) + \prod_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \mu_{\Phi_i}^{(1)}(B) \mu_{\Phi_j}^{(1)}(B) - \left[\sum_{i=1}^{n} \mu_{\Phi_i}^{(1)}(B)\right]^2, \tag{26b}$$

where, using (18c) and (19d), $\mu_{\Phi_i}^{(2)}(B, B) = \mu_{\Phi_i}^{(1)}(B)$. Thus

$$\mathrm{var}_\Phi(B) = \sum_{i=1}^{n} \mu_{\Phi,i}^{(1)}(B) - \sum_{i=1}^{n} \left[\mu_{\Phi,i}^{(1)}(B)\right]^2 \tag{26c}$$
$$= \sum_{i=1}^{n} \mu_{\Phi,i}^{(1)}(B)\left(1 - \mu_{\Phi,i}^{(1)}(B)\right). \tag{26d}$$

Interestingly, the description of the multi-Bernoulli through its counting measure (21) rather than its PGFl (22) provides an alternative and more direct construction of its regional statistics. Indeed, the random variable $N_\Phi(B)$ being the sum of the independent random variables $N_{\Phi_i}(B)$, using the linearity of the expectation operator yields directly the first moment measure of the compound process:

$$\mu_\Phi^{(1)}(B) = \mathbb{E}[N_\Phi(B)] \tag{27a}$$

$$= \sum_{i=1}^{n} \mathbb{E}[N_{\Phi_i}(B)] \tag{27b}$$

$$= \sum_{i=1}^{n} \mu_{\Phi_i}^{(1)}(B) \tag{27c}$$

Likewise, using the Bienaymé formula [67] for the variance of the sum of independent random variables, the variance of the superposed process is

$$\mathrm{var}_\Phi(B) = \sum_{i=1}^{n} \mathrm{var}_{\Phi_i}(B) \tag{28a}$$

$$= \sum_{i=1}^{n} \mu_{\Phi_i}^{(1)}(B)\left(1 - \mu_{\Phi_i}^{(1)}(B)\right). \tag{28b}$$

Note that the simplicity of the alternative approach (27), (28) holds to the facts that 1) we have access to the expression of the counting measure of the compound process, and 2) the expression is reduced to a sum of independent variables. Retrieving the moment measures from *any* point process through the differentiation of its Laplace functional (11), (12) is more adapted to the study of practical multi-object filters. Indeed, under the assumptions of the usual multi-object filters, the structure of the multi-object Bayesian filtering equations (2), (3) allows the explicit construction of the Laplace functional $\mathscr{L}_{\Phi_{k|k}}$ of the target process. In particular, this method led to the extraction of the regional statistics for the PHD filters in [11].

Despite their obvious advantage for the construction of multi-object filters and the extraction of regional statistics, the applicability of the Laplace functional and the PGFl to the study of multi-Bernoulli point processes is limited. From (23) we see that the Laplace functional of the compound process, built as simple product, is *symmetrical* with respect to the Bernoulli processes Φ_i, even though these processes are described by *individual* information carried by the *potentially distinct* functionals $\mathscr{L}_{\Phi_i}$. One must proceed with care when functionals are differentiated for the derivation of a multi-object filter based on a multi-Bernoulli target process, for the order of the Bernoulli processes is not carried through the differentiation of a product such as (23). Recent works [57,61,62] have introduced the notion of *labelled* multi-Bernoulli process in order to produce filters propagating specific information about individual targets, and thus able to reconstruct track histories.

6.3 Regional mean and variance: interpretation

Each component Φ_i of the multi-Bernoulli process describes a single target with an associated probability of existence, and this influences the expression of the expected number of target within some region $B \in \mathbf{B}_{\mathscr{X}}$. For a given probability of existence P_e, the expression of the first moment measure (18c) shows that the expected target number is:

- Minimal if the target lies within B *almost never*, because $\int 1_B(x) p_{\mathrm{s},i}(dx) = 0$;
- Maximal if the target lies within B *almost surely*, because $\int 1_B(x) p_{\mathrm{s},i}(dx) = 1$.

The expected target number is then scaled with the probability of existence. In particular, the extrema $\mu_\Phi^{(1)}(B) = 1$ is reached if and only if the target exists almost surely ($P_\mathrm{e} = 1$) *and* it lies inside within B almost surely. In any case, for any region $B \in \mathbf{B}_{\mathscr{X}}$ (including, in particular, the case $B = \mathscr{X}$) the expected target number may not exceed one.

From the expression of the variance (20b) it is straightforward to see that the variance in B is:

- Minimal (i.e. 0) if $\mu_{\Phi_i}^{(1)}(B) = 0$ or $\mu_{\Phi_i}^{(1)}(B) = 1$;
- Maximal (i.e. 0.25) if $\mu_{\Phi_i}^{(1)}(B) = 0.5$.

Quite intuitively, it shows that the uncertainty in the expected target number is minimal if the evidence of target presence within B reaches one of its two extremas, and is maximal when the absence or presence of a target within B are equally probable. An interesting case arises when the target exists almost surely somewhere in the whole state space, i.e. when $P_\mathrm{e} = 1$. In this case, the sole source of uncertainty is the localization of the target in the surveillance scene, and the expected target number in B is:

- Minimal if the target lies within B almost never *or* almost surely, because $\int 1_B(x) p_{\mathrm{s},i}(dx) = 0$ or $\int 1_B(x) p_{\mathrm{s},i}(dx) = 1$;
- Maximal if the target lies within or outside of B *with equal probability*, because $\int 1_B(x) p_{\mathrm{s},i}(dx) = 0.5$.

Unsurprisingly, each component process Φ_i contributes independently to the regional statistics of the compound process evaluated in B. In particular, the uncertainties in the presence of each target within B add together to form the global uncertainty in the total target number within the same region.

7. Assessment of multi-object filters: analysis of the estimation in cardinality

The study of a point process through its regional statistics produces intuitive and easily interpretable results by shifting the focus from the abstract point process state $E_{\mathscr{X}}$ to a given subset B in $\mathbf{B}_{\mathscr{X}}$. Suppose that B designs a region of the surveillance scene of particular interest (e.g. the surroundings of a building), the regional variance $\mathrm{var}_\Phi(B)$ quantifies the certainty of the filter it its *own estimation* of the number of targets evolving within B. As illustrated in [11], choosing smaller regions B as the immediate surroundings of targets can provide some insight on the ability of a filter to resolve close targets.

The objective of this section is to explain the analysis of the variance in target number of a given multi-object filter. Note that this analysis can be conducted in any region of interest in the state space. We distinguish and study four cases of means and variances from typical multi-object filters. Note that these results are not from actual filters but they are meant to represent different classes of methods for multi-object estimation, i.e. optimal, well-behaved, over-confident, and inaccurate filters.

The Kalman filter is considered as an optimal single-object filter when the observation and motion models are linear and Gaussian. Similarly, a multi-object filter can be thought as being optimal when no approximations are required under given assumptions. The Bayes' filter is an example of optimal multi-object filter under the assumptions that targets are independent of each other and that each of them generates no more than one observation per scan. The mean and variance of an optimal multi-object filter (Figure 1a) are considered as references for other multi-object filters. Note that the mean of the filter is useful as it depicts the best achievable object-detection performance, which can be very different from the ground truth (see Figure 1a at times $t = 40$ and $t = 80$), whereas the use of the Kalman filter mean for comparison purposes is seldom. We can now consider sub-optimal filters, for which approximations are made, and compare them to the optimal filter.

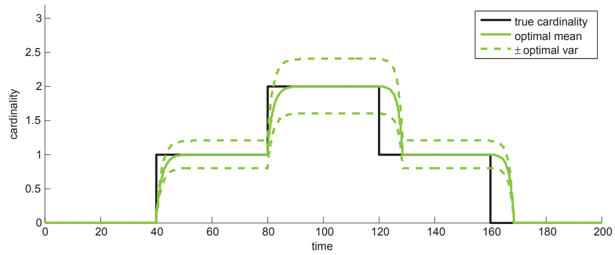
We first consider a multi-object filter that is representative of the class of filters based on realistic approximations. The mean and variance of this filter are depicted in Figure 1b and compared against the mean of the optimal filter. Also, the variances of these two filters can be found in Figure 1e. We can see in Figure 1b that this filter is less reactive to object appearance and disappearance. However, this performance impoverishment is taken into account in the variance, which is consequently larger than the one of the optimal filter. This filter is then seen as a model of well-behaved sub-optimal filter.

The mean and variance of an over-confident multi-object filter are depicted in Figure 1c. This kind of filter is characterised by a larger error in the estimate than what is represented by the variance. It is then not advisable to conclude on the real number of detectable targets as the actual error could be larger than, say, the $\pm 3\sigma$ range (see Figure 1e). It is usual to say that the estimate of the number of targets is inconsistent.

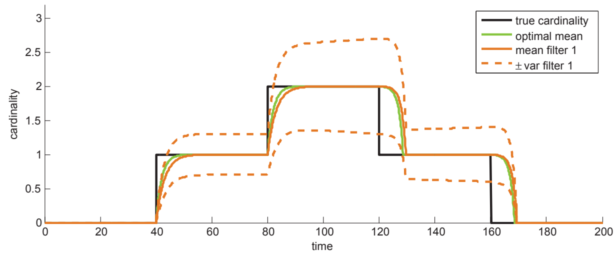
The last typical case consists of a multi-object filter displaying a large variance as in Figure 1d. Whereas the mean of the estimated cardinality is close to the one of the well-behaved filter, the large values of the variance show that this filter relies on stronger approximations, therefore making the estimation less reliable.

8. Conclusion

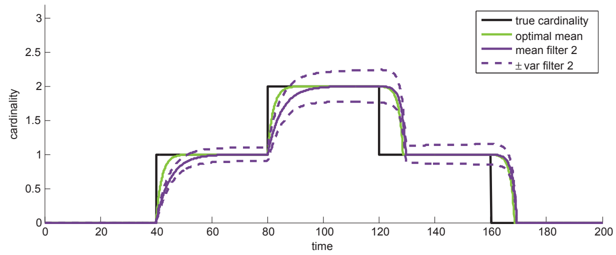
Recent work in multi-object filtering shows that the variance in multi-object estimators can be computed to give information about the uncertainty in the number of targets in any particular region of the state space to aid sensor management and resource allocation. This work computes the target-number variance for the multi-Bernoulli distribution, a common representation of the multi-target state estimate in the tracking literature in which each target is represented with single-object posterior distribution and an existence probability. An illustration of how to interpret using the variance for performance assessment is given through an analysis of optimal, well-behaved, over-confident, and inaccurate filter results.



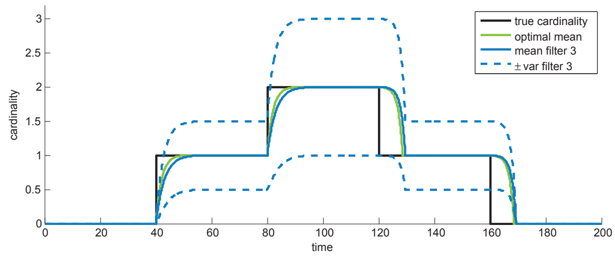
(a) Optimal filter



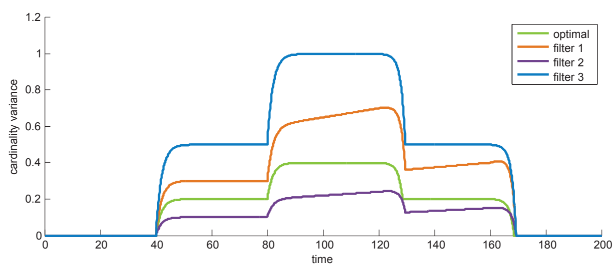
(b) Well-behaved filter



(c) Over-confident filter



(d) Inaccurate filter



(e) Comparison of variances

Figure 1: Mean and variance of the posterior distribution of typical multi-object filters.

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