Iterative Algorithms for Polynomial Eigenvalue Decomposition and Applications

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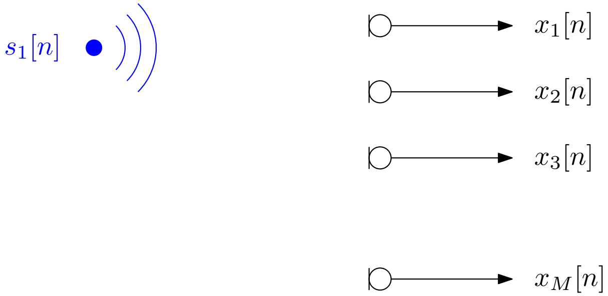
Presentation Overview

1. Overview
2. Eigenvalue (EVD) and singular value decomposition (SVD)
3. Narrowband source separation
4. Broadband problem
5. Polynomial matrices and basic operations
6. Polynomial eigenvalue decomposition algorithms
   - 6.1 sequential best rotation (SBR2)
   - 6.2 sequential matrix diagonalisation (SMD)
7. Applications
   - 7.1 broadband / polynomial subspace decomposition
   - 7.2 polynomial MUSIC
8. Summary

Narrowband Source Model

- Scenario with sensor array and far-field sources:



- for the narrowband case, the source signals arrive with delays, expressed by phase shifts in a steering vector
- data model:

$$\mathbf{x}[n] =$$

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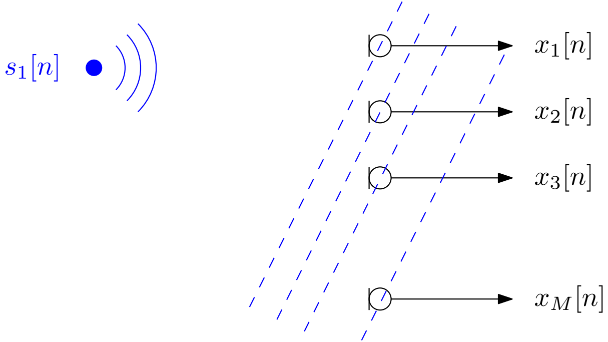

- for the narrowband case, the source signals arrive with delays, expressed by phase shifts in a steering vector $\mathbf{s}_1$
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$$\mathbf{x}[n] = s_1[n] \cdot \mathbf{s}_1$$

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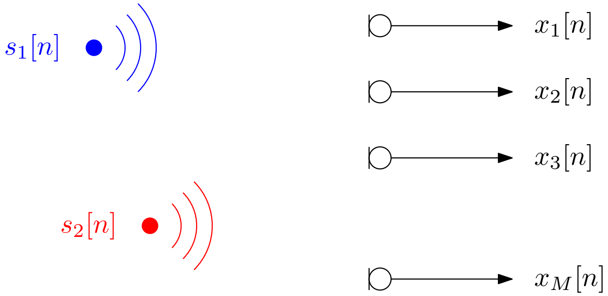

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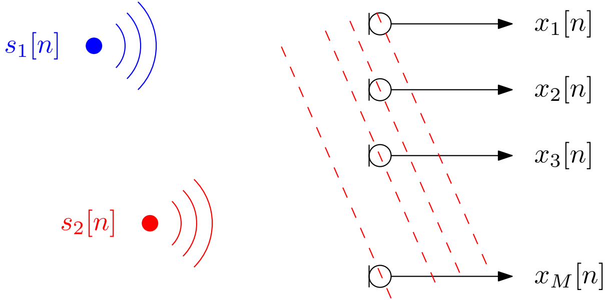

- for the narrowband case, the source signals arrive with delays, expressed by phase shifts in a steering vector $\mathbf{s}_1$, $\mathbf{s}_2$
- data model:

$$\mathbf{x}[n] = s_1[n] \cdot \mathbf{s}_1 + s_1[n] \cdot \mathbf{s_2}$$

Narrowband Source Model

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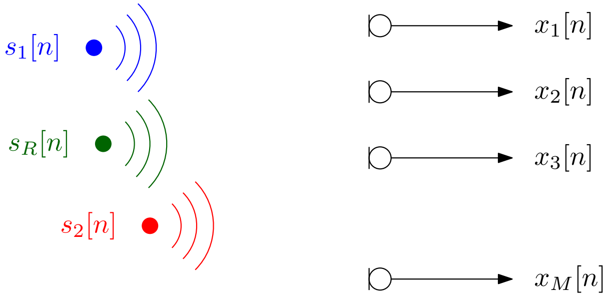

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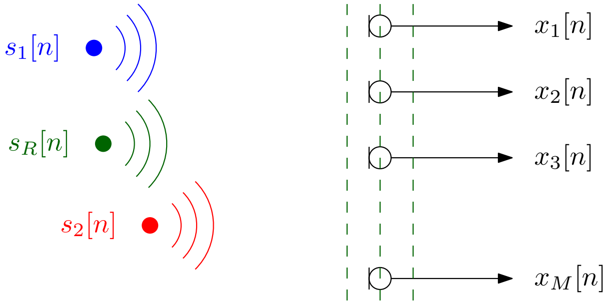

- for the narrowband case, the source signals arrive with delays, expressed by phase shifts in a steering vector $\mathbf{s}_1$, $\mathbf{s}_2$, $\ldots \mathbf{s}_R$;
- data model:

$$\mathbf{x}[n] = s_1[n] \cdot \mathbf{s_1} + s_1[n] \cdot \mathbf{s_2} + \cdots + s_R[n] \cdot \mathbf{s}_R = \sum_{r=1}^{R} s_r[n] \cdot \mathbf{s}_r$$

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Steering Vector

- A signal $s[n]$ arriving at the array can be characterised by the delays of its wavefront (neglecting attenuation):

$$\begin{bmatrix} x_0[n] \\ x_1[n] \\ \vdots \\ x_{M-1}[n] \end{bmatrix} = \begin{bmatrix} s[n-\tau_0] \\ s[n-\tau_1] \\ \vdots \\ s[n-\tau_{M-1}] \end{bmatrix} = \begin{bmatrix} \delta[n-\tau_0] \\ \delta[n-\tau_1] \\ \vdots \\ \delta[n-\tau_{M-1}] \end{bmatrix} * s[n] \circ\!\!-\!\!\bullet\ \mathbf{a}_\vartheta(z) S(z)$$

- if evaluated at a narrowband normalised angular frequency Ω_i, the time delays τ_m in the broadband steering vector $\mathbf{a}_\vartheta(z)$ collapse to phase shifts in the narrowband steering vector $\mathbf{a}_{\vartheta,\Omega_i}$,

$$\mathbf{a}_{\vartheta,\Omega_i} = \mathbf{a}_\vartheta(z)|_{z=e^{j\Omega_i}} = \begin{bmatrix} e^{-j\tau_0\Omega_i} \\ e^{-j\tau_1\Omega_i} \\ \vdots \\ e^{-j\tau_{M-1}\Omega_i} \end{bmatrix} .$$

Data and Covariance Matrices

- A data matrix $\mathbf{X} \in \mathbb{C}^{M\times L}$ can be formed from L measurements:

$$\mathbf{X} = \left[\begin{array}{cccc} \mathbf{x}[n] & \mathbf{x}[n+1] & \ldots & \mathbf{x}[n+L-1] \end{array} \right]$$

- assuming that all $x_m[n]$, $m = 1, 2, \ldots M$ are zero mean, the (instantaneous) data covariance matrix is

$$\mathbf{R} = \mathcal{E}\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n]\} \approx \frac{1}{L}\mathbf{X}\mathbf{X}^{\mathrm{H}}$$

  where the approximation assumes ergodicity and a sufficiently large L;
- Problem: can we tell from $\mathbf{X}$ or $\mathbf{R}$ (i) the number of sources and (ii) their orgin / time series?
- w.r.t. Jonathon Chamber's introduction, we here only consider the underdetermined case of more sensors than sources, $M \geq K$, and generally $L \gg M$.

SVD of Data Matrix

- Singular value decomposition of $\mathbf{X}$:

$$\mathbf{X} = \mathbf{U} \; \boldsymbol{\Sigma} \; \mathbf{V}^{\mathrm{H}}$$

- unitary matrices $\mathbf{U} = [\mathbf{u}_1 \ldots \mathbf{u}_M]$ and $\mathbf{V} = [\mathbf{v}_1 \ldots \mathbf{v}_L]$;
- diagonal $\boldsymbol{\Sigma}$ contains the real, positive semidefinite singular values of $\mathbf{X}$ in descending order:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & & 0 & \sigma_M & 0 & \ldots & 0 \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_M \geq 0$.

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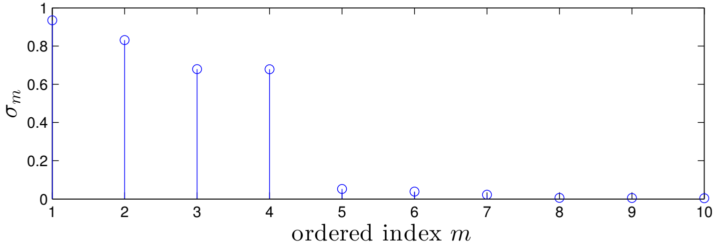
Singular Values

- If the array is illuminated by $R \leq M$ linearly independent sources, the rank of
  the data matrix is
  $$\text{rank}\{\mathbf{X}\} = R$$
- only the first R singular values of $\mathbf{X}$ will be non-zero;
- in practice, noise often will ensure that $\text{rank}\{\mathbf{X}\} = M$, with $M - R$ trailing singular values that define the noise floor:



- therefore, by thresholding singular values, it is possible to estimate the number of linearly independent sources R.

Subspace Decomposition

- If $\text{rank}\{\mathbf{X}\} = R$, the SVD can be split:

$$\mathbf{X} = [\mathbf{U}_s \;\; \mathbf{U}_n] \begin{bmatrix} \boldsymbol{\Sigma}_s & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_n \end{bmatrix} \begin{bmatrix} \mathbf{V}_s^{\mathrm{H}} \\ \mathbf{V}_n^{\mathrm{H}} \end{bmatrix}$$

- with $\mathbf{U}_s \in \mathbb{C}^{M\times R}$ and $\mathbf{V}_s^{\mathrm{H}} \in \mathbb{C}^{R\times L}$ corresponding to the R largest singular values;
- $\mathbf{U}_s$ and $\mathbf{V}_s^{\mathrm{H}}$ define the signal-plus-noise subspace of $\mathbf{X}$:

$$\mathbf{X} = \sum_{m=1}^{M} \sigma_m \mathbf{u}_m \mathbf{v}_m^{\mathrm{H}} \approx \sum_{m=1}^{R} \sigma_m \mathbf{u}_m \mathbf{v}_m^{\mathrm{H}}$$

- the complements $\mathbf{U}_n$ and $\mathbf{V}_n^{\mathrm{H}}$,

$$\mathbf{U}_s^{\mathrm{H}}\mathbf{U}_n = \mathbf{0} \qquad , \qquad \mathbf{V}_s\mathbf{V}_n^{\mathrm{H}} = \mathbf{0}$$

define the noise-only subspace of $\mathbf{X}$.

SVD via Two EVDs

- Any Hermitian matrix $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$ allows an eigenvalue decomposition

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{H}}$$

  with $\mathbf{Q}$ unitary and the eigenvalues in $\mathbf{\Lambda}$ real valued and positive semi-definite;
- postulating $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}$, therefore:

$$\mathbf{X}\mathbf{X}^{\mathrm{H}} = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}})(\mathbf{V}\mathbf{\Sigma}^{\mathrm{H}}\mathbf{U}^{\mathrm{H}}) = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{H}} \quad (1)$$

$$\mathbf{X}^{\mathrm{H}}\mathbf{X} = (\mathbf{V}\mathbf{\Sigma}^{\mathrm{H}}\mathbf{U}^{\mathrm{H}})(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{H}}) = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathrm{H}} \quad (2)$$

- (ordered) eigenvalues relate to the singular values: $\lambda_m = \sigma_m^2$;
- the covariance matrix $\mathbf{R} = \frac{1}{L}\mathbf{X}\mathbf{X}$ has the same rank as the data matrix $\mathbf{X}$, and with $\mathbf{U}$ provides access to the same spatial subspace decomposition.

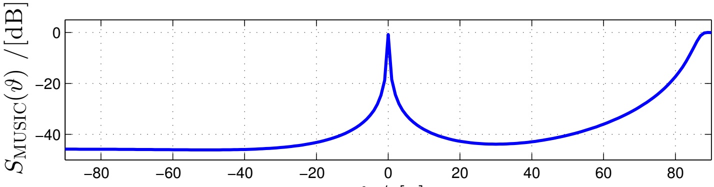
Narrowband MUSIC Algorithm

- EVD of the narrowband covariance matrix identifies signal-plus-noise and noise-only subspaces

$$\mathbf{R} = [\mathbf{U}_s \;\; \mathbf{U}_n] \begin{bmatrix} \mathbf{\Lambda}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_n \end{bmatrix} \begin{bmatrix} \mathbf{U}_s^{\mathrm{H}} \\ \mathbf{U}_n^{\mathrm{H}} \end{bmatrix}$$

- scanning the signal-plus-noise subspace could only help to retrieve sources with orthogonal steering vectors;
- therefore, the multiple signal classification (MUSIC) algorithm scans the noise-only subspace for minima, or maxima of its reciprocal

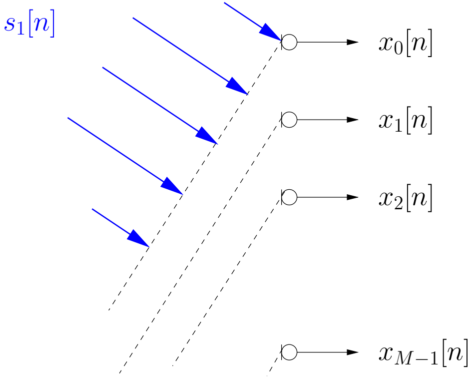
$$S_{\mathrm{MUSIC}}(\vartheta) = \frac{1}{\|\mathbf{U}_n \mathbf{a}_{\vartheta,\Omega_i}\|_2^2}$$



Narrowband Source Separation

- Via SVD of the data matrix $\mathbf{X}$ or EVD of the covariance matrix $\mathbf{R}$, we can determine the number of linearly independent sources R;
- using the subspace decompositions offered by EVD/SVD, the directions of arrival can be estimated using e.g. MUSIC;
- based on knowledge of the angle of arrival, beamforming could be applied to $\mathbf{X}$ to extract specific sources;
- overall: EVD (and SVD) can play a vital part in narrowband source separation;
- what about broadband source separation?

Broadband Array Scenario



- Compared to the narrowband case, time delays rather than phase shifts bear information on the direction of a source.

Broadband Steering Vector

- A signal $s[n]$ arriving at the array can be characterised by the delays of its wavefront (neglecting attenuation):

$$\begin{bmatrix} x_0[n] \\ x_1[n] \\ \vdots \\ x_{M-1}[n] \end{bmatrix} = \begin{bmatrix} s[n-\tau_0] \\ s[n-\tau_1] \\ \vdots \\ s[n-\tau_{M-1}] \end{bmatrix} = \begin{bmatrix} \delta[n-\tau_0] \\ \delta[n-\tau_1] \\ \vdots \\ \delta[n-\tau_{M-1}] \end{bmatrix} * s[n] \circ\!\!-\!\!\bullet\ \mathbf{a}_\vartheta(z) S(z)$$

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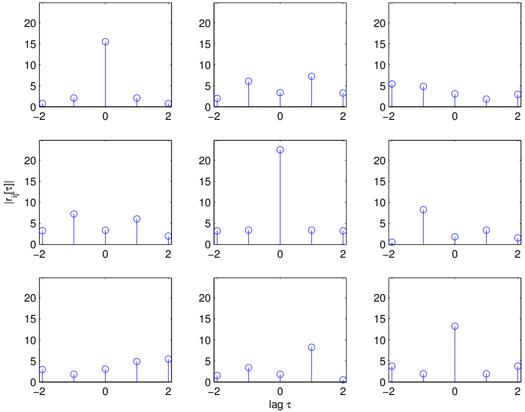
Space-Time Covariance Matrix

- If delays must be considered, the (space-time) covariance matrix must capture the lag τ:

$$\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}[n] \cdot \mathbf{x}^{\mathrm{H}}[n-\tau]\}$$

- $\mathbf{R}[\tau]$ contains auto- and cross-correlation sequences:



Cross Spectral Density Matrix

- z-transform of the space-time covariance matrix is given by

$$\mathbf{R}[\tau] = \mathcal{E}\{\mathbf{x}_n \mathbf{x}_{n-\tau}^{\mathrm{H}}\} \quad \circ\!\!-\!\!\bullet \quad \boldsymbol{R}(z) = \sum_l S_l(z)\mathbf{a}_{\vartheta_l}(z)\tilde{\mathbf{a}}_{\vartheta_l}(z) + \sigma_N^2 \mathbf{I}$$

  with ϑ_l the direction of arrival and $S_l(z)$ the PSD of the lth source;
- $\boldsymbol{R}(z)$ is the cross spectral density (CSD) matrix;
- the instantaneous covariance matrix (no lag parameter τ)

$$\mathbf{R} = \mathcal{E}\{\mathbf{x}_n \mathbf{x}_n^{\mathrm{H}}\} = \mathbf{R}[0]$$

CSD Matrix Properties

- The CSD matrix $\boldsymbol{R}(z)$ is a matrix polynomial or polynomial with matrix-valued coefficients:

$$\boldsymbol{R}(z) = \cdots + \mathbf{R}_{-2}z^{2} + \mathbf{R}_{-1}z^{1} + \mathbf{R}_{0} + \mathbf{R}_{1}z^{-1} + \mathbf{R}_{2}z^{-2} + \ldots$$

- the symmetry of the cross-correlation sequences $r_{xy}[\tau] = r^{*}_{yx}[-\tau]$ is reflected in the CSD matrix $\boldsymbol{R}(z)$:

$$\mathbf{R}_{\tau} = \mathbf{R}^{\mathrm{H}}_{-\tau}$$

- therefore with the parahermitian operator $\tilde{\{\cdot\}}$

$$\boldsymbol{R}(z) = \boldsymbol{R}^{\mathrm{H}}(z^{-1}) = \tilde{\boldsymbol{R}}(z)$$

- a matrix fulfilling $\boldsymbol{R}(z) = \tilde{\boldsymbol{R}}(z)$ is called a parahermitian matrix.

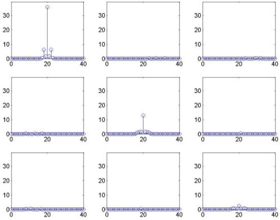
Polynomial Eigenvalue Decomposition

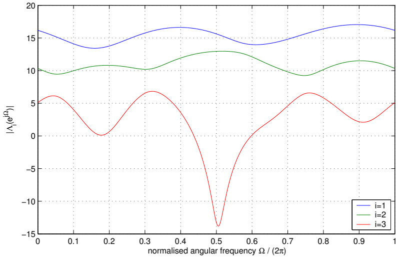
[McWhirter *et al.*, IEEE TSP 2007]

- Polynomial EVD of the CSD matrix

$$\boldsymbol{R}(z) = \boldsymbol{Q}(z)\ \boldsymbol{\Lambda}(z)\ \tilde{\boldsymbol{Q}}(z)$$

- with paraunitary $\mathbf{Q}(z)$, s.t. $\mathbf{Q}(z)\tilde{\mathbf{Q}}(z) = \mathbf{I}$;
- diagonalised and spectrally majorised $\boldsymbol{\Lambda}(z)$:





- $\boldsymbol{Q}(z)$ can be FIR of sufficiently high order [Icart & Comon 2012]

Iterative PEVD Algorithms

- Second order sequential best rotation (SBR2, McWhirter 2007);
- iterative approach based on an elementary paraunitary operation:

$$
\begin{aligned}
\boldsymbol{S}_0(z) &= \boldsymbol{R}(z) \\
&\vdots \\
\boldsymbol{S}_{i+1}(z) &= \tilde{\boldsymbol{H}}_{i+1}(z)\boldsymbol{S}_{i+1}(z)\boldsymbol{H}_{i+1}(z)
\end{aligned}
$$

- $\boldsymbol{H}_i(z)$ is an elementary paraunitary operation, which at the ith step eliminates the largest off-diagonal element in $\boldsymbol{s}_{i-1}(z)$;
- stop after L iterations:

$$
\hat{\boldsymbol{\Lambda}}(z) = \boldsymbol{S}_L(z) \qquad , \qquad \boldsymbol{Q}(z) = \prod_{i=1}^{L} \boldsymbol{H}_i(z)
$$

- sequential matrix diagonalisation (SMD) and
- multiple-shift SMD (MS-SMD) will follow the same scheme ...

Elementary Paraunitary Operation

- An elementary paraunitary matrix [Vaidyanathan] is defined as

$$\boldsymbol{H}_i(z) = \mathbf{I} - \mathbf{v}_i\mathbf{v}_i^{\mathrm{H}} + z^{-1}\mathbf{v}_i\mathbf{v}_i^{\mathrm{H}}$$

- we utilise a different definition:

$$\boldsymbol{H}_i(z) = \boldsymbol{D}_i(z)\mathbf{Q}_i$$

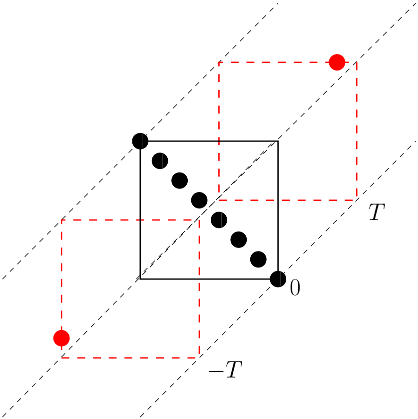
- $\boldsymbol{D}_i(z)$ is a delay matrix:

$$\boldsymbol{D}_i(z) = \mathrm{diag}\{1 \; \ldots \; 1 \; z^{-\tau} \; 1 \; \ldots \; 1\}$$

- $\mathbf{Q}_i(z)$ is a Givens rotation.

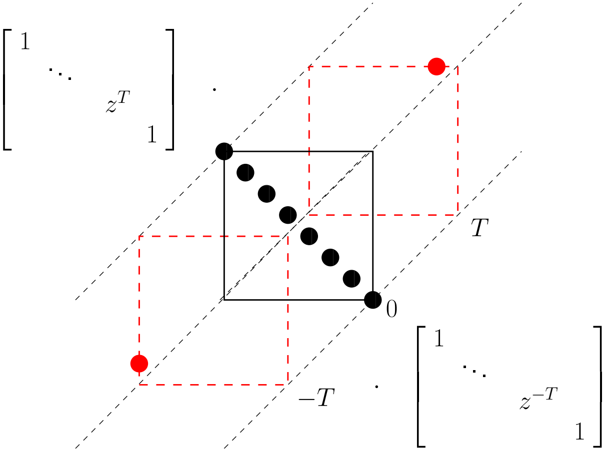
Sequential Best Rotation Algorithm (McWhirter)

- At iteration i, consider $\boldsymbol{S}_{i-1}(z) \circ\!\!-\!\!\bullet \mathbf{S}_{i-1}[\tau]$



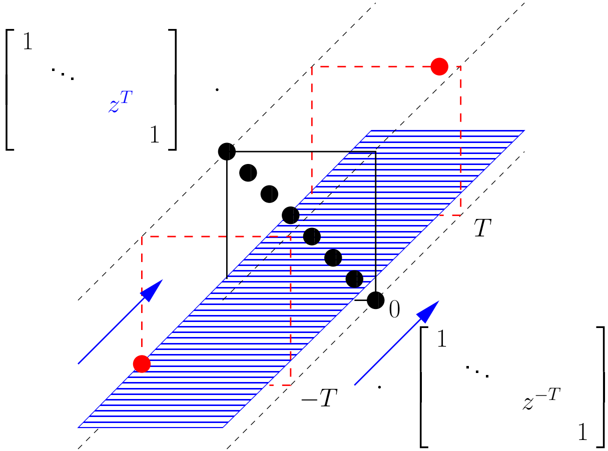
Sequential Best Rotation Algorithm (McWhirter)

- $\tilde{\boldsymbol{D}}_i(z)\boldsymbol{S}_{i-1}(z)\boldsymbol{D}_i(z)$



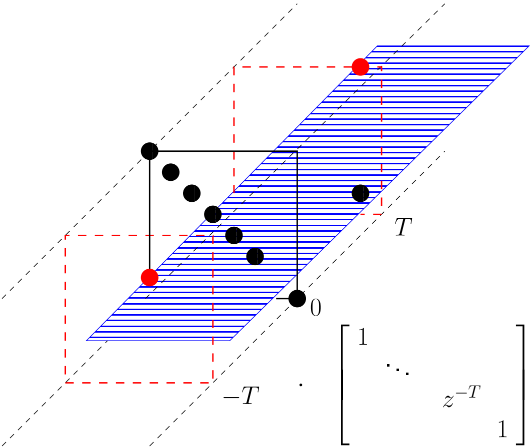
Sequential Best Rotation Algorithm (McWhirter)

- $\tilde{\boldsymbol{D}}_i(z)$ advances a row-slice of $\boldsymbol{S}_{i-1}(z)$ by T



Sequential Best Rotation Algorithm (McWhirter)

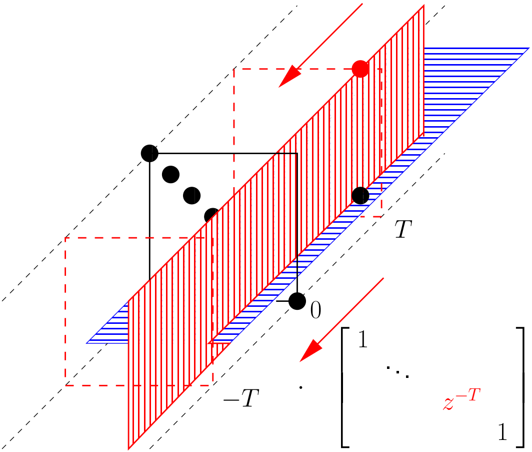
- the off-diagonal element at $-T$ has now been translated to lag zero



$$\cdot \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & z^{-T} & \\ & & & 1 \end{bmatrix}$$

Sequential Best Rotation Algorithm (McWhirter)

- $\mathbf{D}_i(z)$ delays a column-slice of $\boldsymbol{S}_{i-1}(z)$ by T



Sequential Best Rotation Algorithm (McWhirter)

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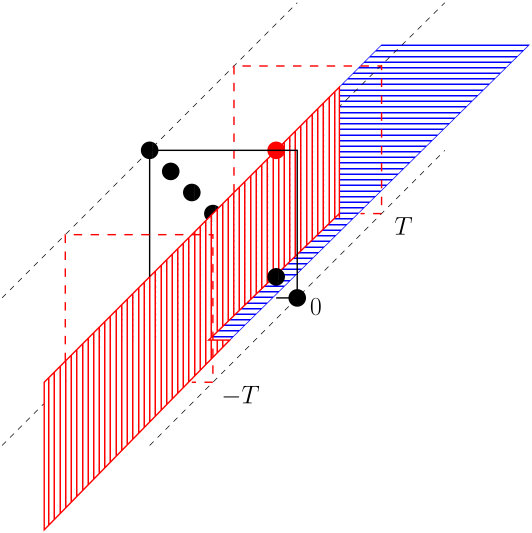
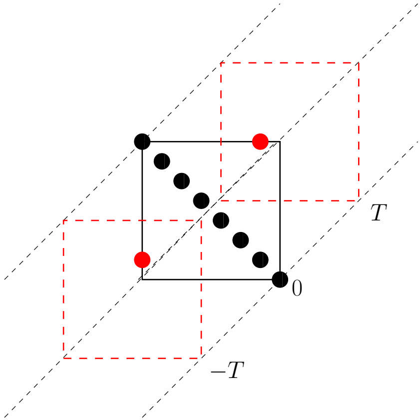


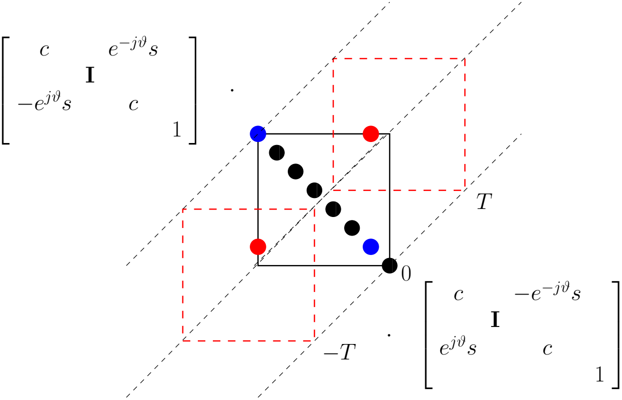
Sequential Best Rotation Algorithm (McWhirter)

- the step $\tilde{\boldsymbol{D}}_i(z)\boldsymbol{S}_{i-1}(z)\boldsymbol{D}_i(z)$ has brought the largest off-diagonal elements to lag 0.



Sequential Best Rotation Algorithm (McWhirter)

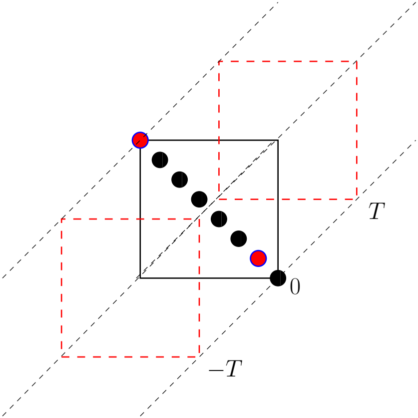
- Jacobi step to eliminate largest off-diagonal elements by $\mathbf{Q}_i$



Sequential Best Rotation Algorithm (McWhirter)

- iteration i is completed, having performed

$$\boldsymbol{S}_i(z) = \mathbf{Q}_i \boldsymbol{D}_i(z) \boldsymbol{S}_{i-1}(z) \tilde{\boldsymbol{D}}_i(z) \tilde{\mathbf{Q}}_i(z)$$



SBR2 Outcome

- At the ith iteration, the zeroing of off-diagonal elements achieved during previous steps may be partially undone;
- however, the algorithm has been shown to converge, transfering energy onto the main diagonal at every step (McWhirter 2007);
- after L iterations, we reach an approximate diagonalisation

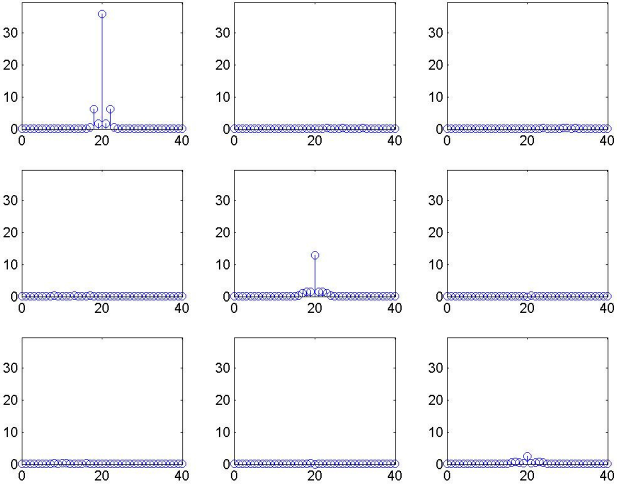
$$\hat{\mathbf{\Lambda}}(z) = \boldsymbol{S}^L(z) = \tilde{\boldsymbol{Q}}(z)\boldsymbol{R}(z)\boldsymbol{Q}(z)$$

with

$$\boldsymbol{Q}(z) = \prod_{i=1}^{L} \boldsymbol{D}_i(z)\mathbf{Q}_i$$

- diagonalisation of the previous 3×3 polynomial matrix . . .

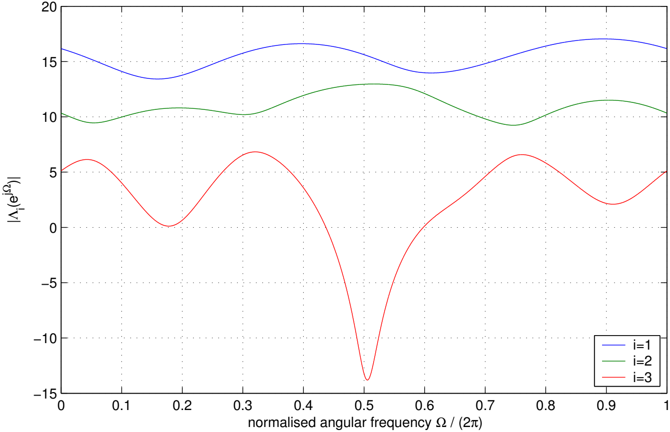
SBR2 Example — Diagonalisation



lag τ

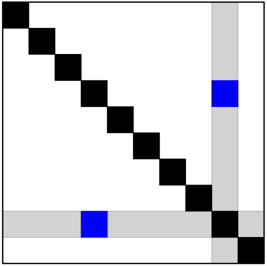
SBR2 Example — Spectral Majorisation

- The on-diagonal elements are spectrally majorised



SBR2 — Givens Rotation

- A Givens rotation eliminates the maximum off-diagonal element once brought onto the lag-zero matrix;
- note I: in the lag-zero matrix, one column and one row are modified by the shift:



- note II: a Givens rotation only affects two columns and two rows in every matrix;
- Givens rotation is relatively low in computational cost!

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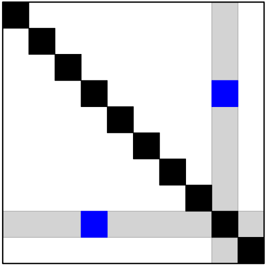
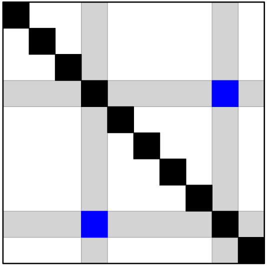
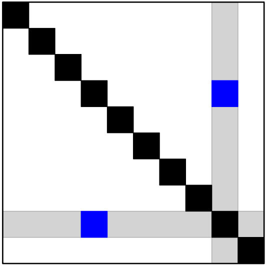




- note II: a Givens rotation only affects two columns and two rows in every matrix;
- Givens rotation is relatively low in computational cost!

Sequential Matrix Diagonalisation (SMD)

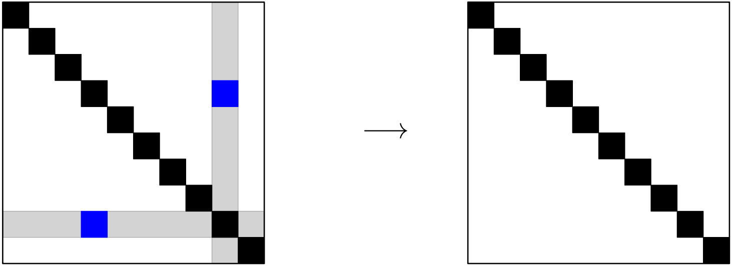
- Main idea — the zero-lag matrix is diagonalised in every step;
- initialisation: diagonalise $\mathbf{R}[0]$ by EVD and apply modal matrix to all matrix coefficients $\longrightarrow \boldsymbol{S}_0$;
- at the ith step as in SBR2, the maximum element (or column with max. norm) is shifted to the lag-zero matrix:



- an EVD is used to re-diagonalise the zero-lag matrix;
- a full modal matrix has to applied at all lags — more costly than SBR2.

Sequential Matrix Diagonalisation (SMD)

- Main idea — the zero-lag matrix is diagonalised in every step;
- initialisation: diagonalise $\mathbf{R}[0]$ by EVD and apply modal matrix to all matrix coefficients $\longrightarrow \boldsymbol{S}_0$;
- at the ith step as in SBR2, the maximum element (or column with max. norm) is shifted to the lag-zero matrix:



- an EVD is used to re-diagonalise the zero-lag matrix;
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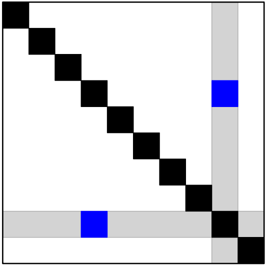
Multiple Shift SMD (SMD)

- SMD converges faster than SBR2 — more energy is transfered per iteration step;
- SMD is more expensive than SBR2 — full matrix multiplication at every lag;
- this cost will not increase further if more columns / rows are shifted into the lag-zero matrix at every iteration



- MS-SMD will transfer yet more off-diagonal energy per iteration;
- because the total energy must remain constant under paraunitary operations, SBR2, SMD and MS-SMD can be proven to converge.

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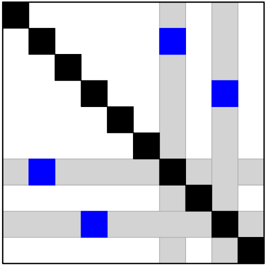

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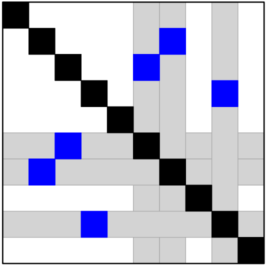

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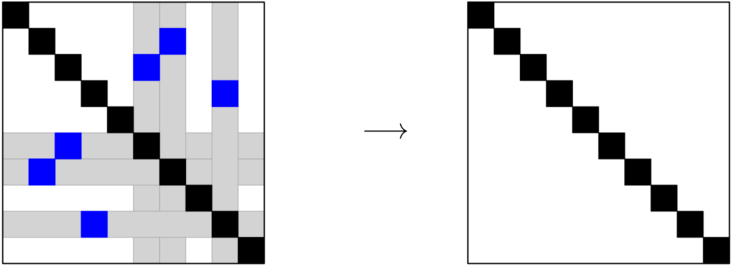

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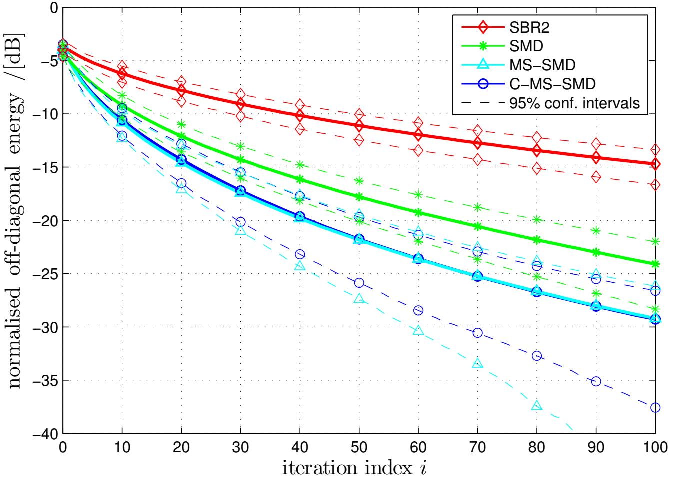
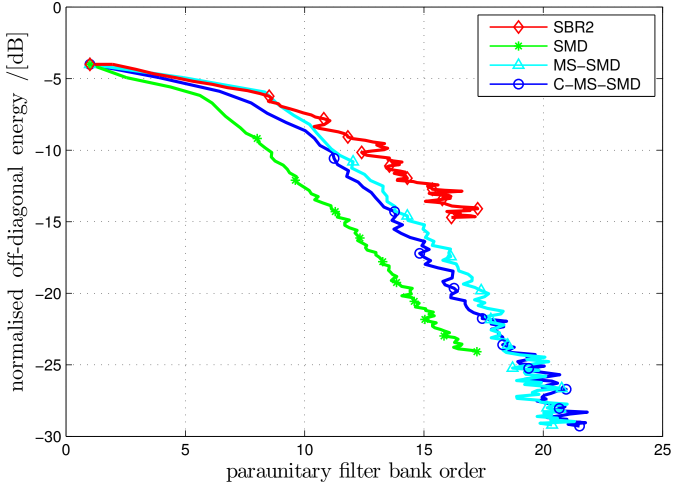
SBR2/SMD/MS-SMD Convergence

- Measuring the remaining normalised off-diagonal energy over an ensemble of space-time covariance matrices:



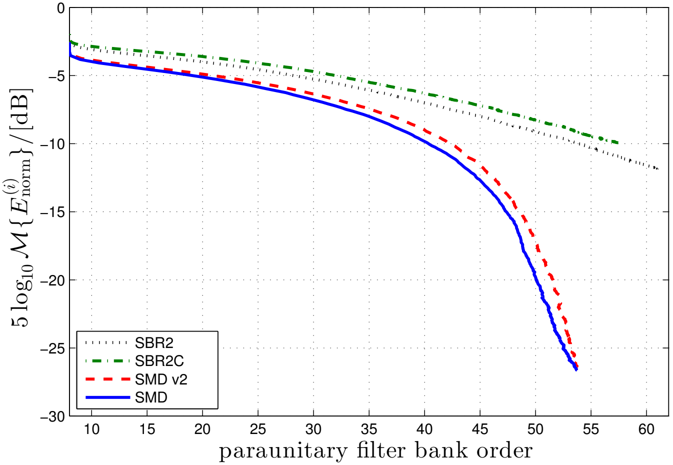
SBR2/SMD/MS-SMD Application Cost 1

- Ensemble average of remaining off-diagonal energy vs. order of paraunitary filter banks to decompose 4x4x16 matrices:



SBR2/SMD/MS-SMD Application Cost 2

- Ensemble average of remaining off-diagonal energy vs. order of paraunitary filter banks to decompose 8x8x64 matrices:



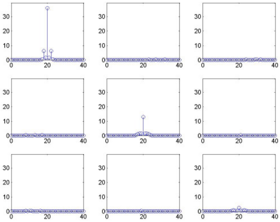
Polynomial MUSIC (PMUSIC)

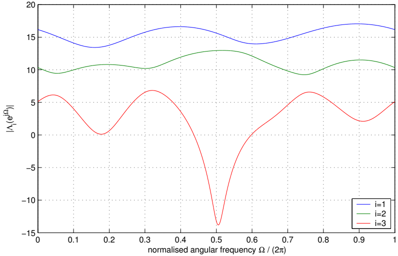
[Alrmah, Weiss, Lambotharan, *EUSIPCO* (2011)]

- Based on the polynomial EVD of the broadband covariance matrix

$$\mathbf{R}(z) \approx \underbrace{[\mathbf{Q}_s(z) \quad \mathbf{Q}_n(z)]}_{\mathbf{Q}(z)} \underbrace{\begin{bmatrix} \mathbf{\Lambda}_s(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_n(z) \end{bmatrix}}_{\mathbf{\Lambda}(z)} \begin{bmatrix} \tilde{\mathbf{Q}}_s(z) \\ \tilde{\mathbf{Q}}_n(z) \end{bmatrix}$$

- paraunitary $\mathbf{Q}(z)$, s.t. $\mathbf{Q}(z)\tilde{\mathbf{Q}}(z) = \mathbf{I}$;
- diagonalised and spectrally majorised $\mathbf{\Lambda}(z)$:





PMUSIC cont'd

- Idea —- scan the polynomial noise-only subspace $Q_n(z)$ with broadband steering vectors

$$\Gamma(z,\vartheta) = \tilde{\mathbf{a}}_\vartheta(z)\tilde{\mathbf{Q}}_n(z)\mathbf{Q}_n(z)\mathbf{a}_\vartheta(z)$$

- looking for minima leads to a spatio-spectral PMUSIC

$$S_{\mathrm{PSS-MUSIC}}(\vartheta,\Omega) = \left(\Gamma(z,\vartheta)|_{z=e^{j\Omega}}\right)^{-1}$$

- and a spatial-only PMUSIC

$$S_{\mathrm{PS-MUSIC}}(\vartheta) = \left(2\pi \oint \Gamma(z,\vartheta)|_{z=e^{j\Omega}} d\Omega\right)^{-1} = \Gamma_\vartheta^{-1}[0]$$

with $\Gamma_\vartheta[\tau] \circ\!\!-\!\!\bullet\, \Gamma(z,\vartheta)$.

Simulation I — Toy Problem

- Linear uniform array with critical spatial and temporal sampling;
- broadband steering vector for end-fire position:

$$\mathbf{a}_{\pi/2}(z) = [1 \;\; z^{-1} \;\; \cdots \;\; z^{-M+1}]^{\mathrm{T}}$$

- covariance matrix

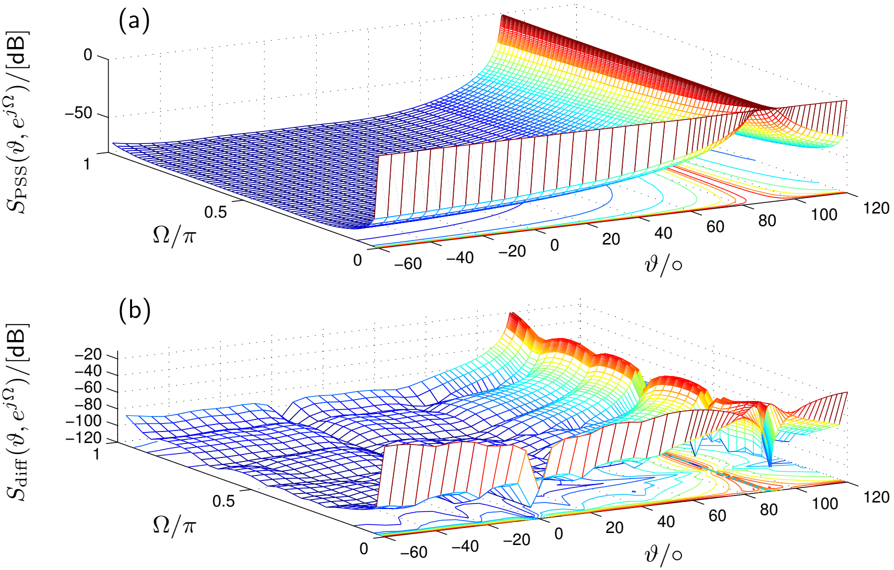
$$\mathbf{R}(z) = \mathbf{a}_{\pi/2}(z)\tilde{\mathbf{a}}_{\pi/2}(z) = \begin{bmatrix} 1 & z^{1} & \ldots & z^{M-1} \\ z^{-1} & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ z^{-M+1} & \ldots & \ldots & 1 \end{bmatrix} .$$

- PEVD (by inspection)

$$\mathbf{Q}(z) = \mathbf{T}_{\mathrm{DFT}}\mathrm{diag}\{1 \;\; z^{-1} \;\; \cdots \;\; z^{-M+1}\} \quad ; \quad \boldsymbol{\Lambda}(z) = \mathrm{diag}\{1 \; 0 \; \cdots \; 0$$

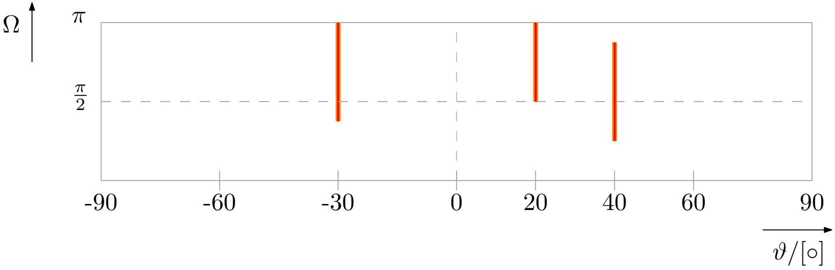
- simulations with $M = 4 \ldots$

Simulation I — PSS-MUSIC



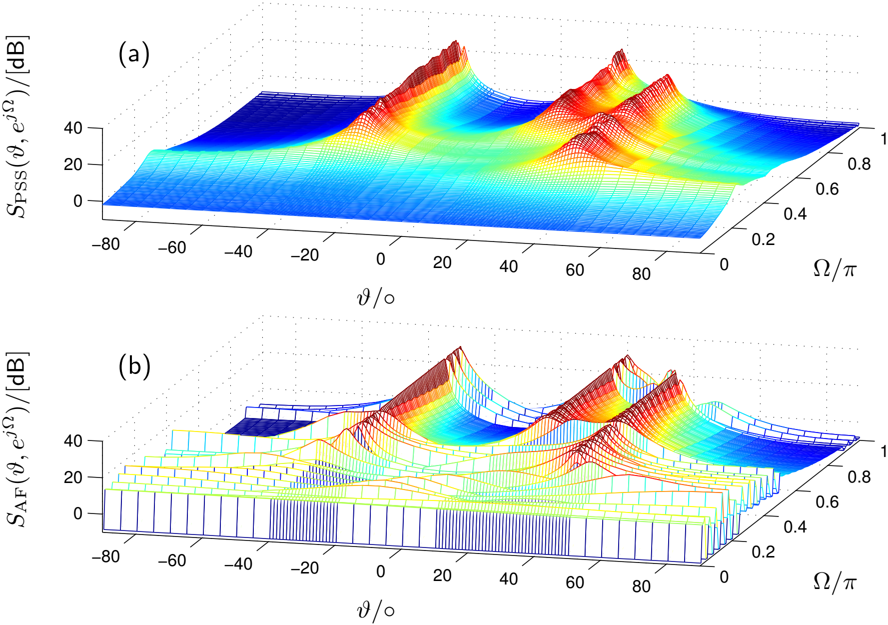
Simulation II

- $M = 8$ element sensor array illuminated by three sources;
- source 1: $\vartheta_1 = -30°$, active over range $\Omega \in [\frac{3\pi}{8};\ \pi]$;
- source 2: $\vartheta_2 = 20°$, active over range $\Omega \in [\frac{\pi}{2};\ \pi]$;
- source 3: $\vartheta_3 = 40°$, active over range $\Omega \in [\frac{2\pi}{8};\ \frac{7\pi}{8}]$; and



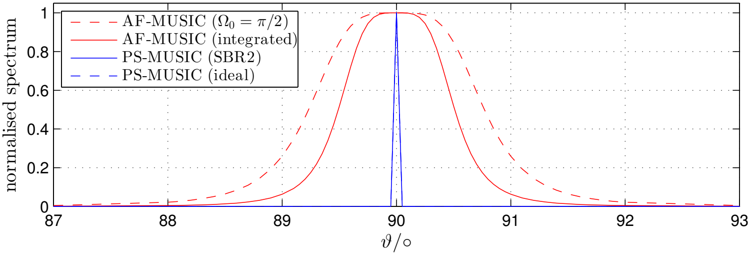
- filter banks as innovation filters, and broadband steering vectors to simulate AoA;
- space-time covariance matrix is estimated from 10^4 samples.

Simulation II — PSS-MUSIC

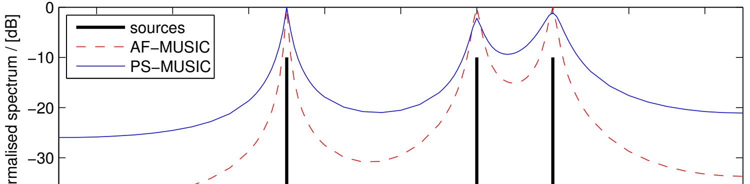


PS-MUSIC Comparison

- Simulation I (toy problem): peaks normalised to unity:



- Simulation II: inaccuracies on PEVD and broadband steering vector



Conclusions

- We have considered the importance of SVD and EVD for narrowband source separation;
- narrowband matrix decomposition real the matrix rank and offer subspace decompositions on which angle-of-arrival estimation alhorithms such as MUSIC can be based;
- broadband problems lead to a space-time covariance or CSD matrix;
- such polynomial matrices cannot be decomposed by standard EVD and SVD;
- a polynomial EVD has been defined;
- iterative algorithms such as SBR2 can be used to approximate the PEVD;
- this permits a number of applications, such as broadband angle of arrival estimation;
- broadband beamforming could then be used to separate broadband sources.

Additional Material

- Papers included on the USB drives:
  1. J.G. McWhirter, P.D. Baxter, T. Cooper, S. Redif, and J. Foster: "An EVD Algorithm for Para-Hermitian Polynomial Matrices," IEEE Transactions on Signal Processing, **55**(5): 2158-2169, May 2007.
  2. S, Redif, J.G. McWhirter, and S. Weiss: "Design of FIR Paraunitary Filter Banks for Subband Coding Using a Polynomial Eigenvalue Decomposition," IEEE Transactions on Signal Processing, **59**(11): 5253-5264, Nov. 2011.
  3. P. Baxter and J.G. McWhirter: "Blind signal separation of convolutive mixtures," Proc. 37th Asilomar Conference on Signals, Systems and Computers, **1**: 124-128, November 2003.
- If interested in trying the PEVD and its iterative algorithms yourself, please e-mail:
  - Jamie Corr (jamie.corr@strath.ac.uk), or
  - Stephan Weiss (stephan.weiss@strath.ac.uk)
- coming soon: Matlab-compatible toolbox with PEVD algorithms.