Optimal and Adaptive Filtering

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Table of Contents

Optimal filter design

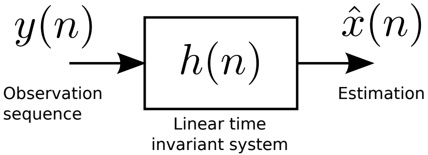


Figure 1: Optimal filtering scenario.

- $y(n)$: Observation related to a **stationary** signal of interest $x(n)$.
- $h(n)$: The impulse response of an LTI estimator.
- $\hat{x}(n)$: Estimate of $x(n)$ given by

$$\hat{x}(n) = h(n) * y(n) = \sum_{i=-\infty}^{\infty} h(i)y(n-i).$$

Optimal filter design

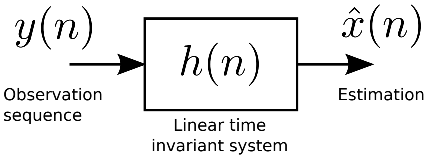


Figure 1: Optimal filtering scenario.

- Find $h(n)$ with the best error performance:

$$e(n) = x(n) - \hat{x}(n) = x(n) - h(n) * y(n)$$

- The error performance is measured by the mean squared error (MSE)

$$\xi = E\left[\left(e(n)\right)^2\right].$$

Optimal filter design

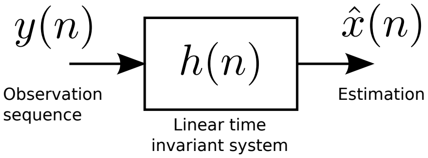


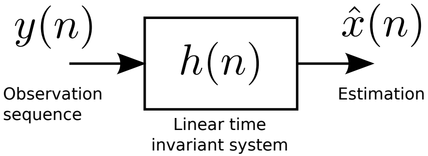
Figure 1: Optimal filtering scenario.

- The MSE is a function of $h(n)$, i.e.,

$$\mathbf{h} = [\cdots, h(-2), h(-1), h(0), h(1), h(2), \cdots]$$

$$\xi(\mathbf{h}) = E\left[\Big(e(n)\Big)^2\right] = E\left[\Big(x(n) - h(n) * y(n)\Big)^2\right].$$

Optimal filter design



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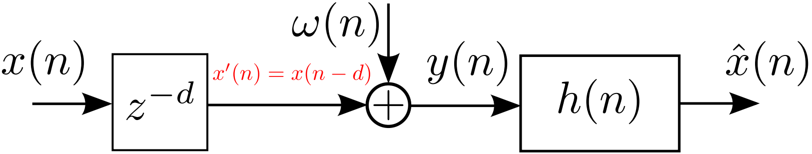
$$\xi(\mathbf{h}) = E\left[\left(e(n)\right)^2\right] = E\left[\left(x(n) - h(n) * y(n)\right)^2\right].$$

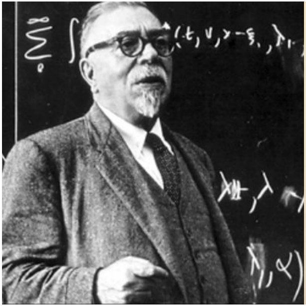
- Thus, optimal filtering problem is

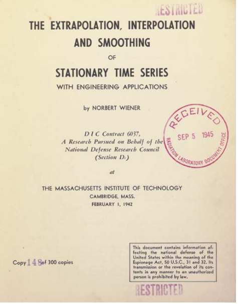
$$\mathbf{h}_{opt} = \arg\min_{\mathbf{h}} \xi(\mathbf{h})$$

Application examples

1) Prediction, interpolation and smoothing of signals







RESTRICTED

THE EXTRAPOLATION, INTERPOLATION AND SMOOTHING OF STATIONARY TIME SERIES

WITH ENGINEERING APPLICATIONS

by NORBERT WIENER

D I C Contract 6037,
A Research Pursued on Behalf of the National Defense Research Council (Section D2)

at

THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Application examples

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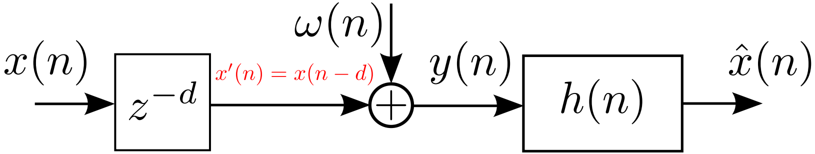
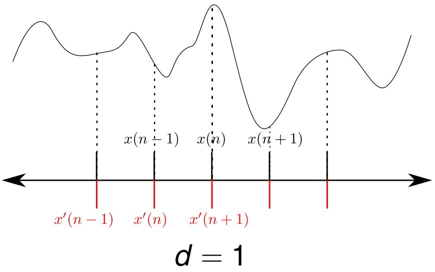
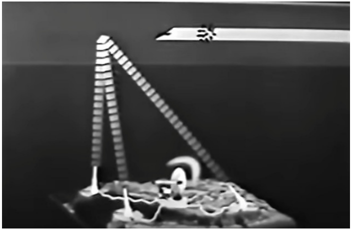


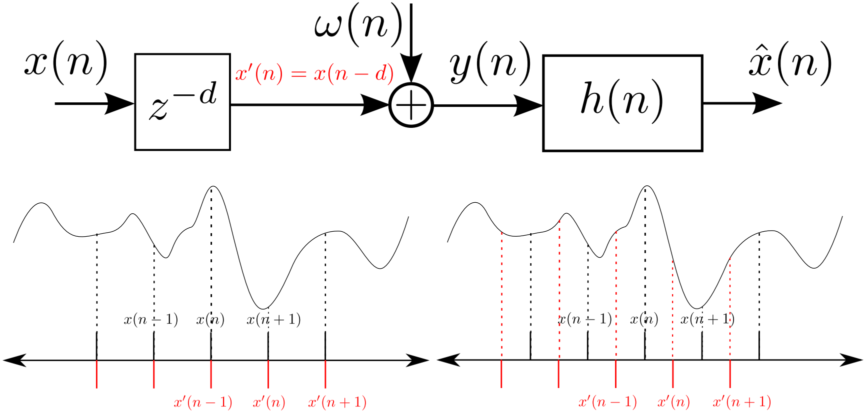




- Prediction for anti-aircraft fire control.

Application examples

1) Prediction, interpolation and smoothing of signals



$d = -1$ (smoothing) $\quad$ $d = -1/2$ (interpolation)

- Signal denoising applications, estimation of missing data points.

Application examples

2) System identification

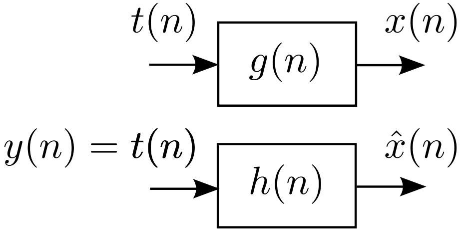


Figure 2: System identification using a training sequence $t(n)$ from an ergodic and stationary ensemble.

Application examples

2) System identification

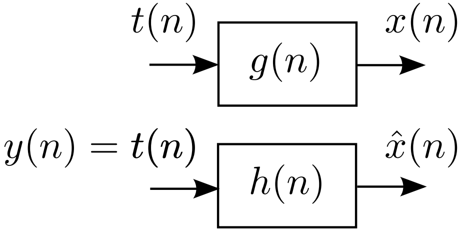
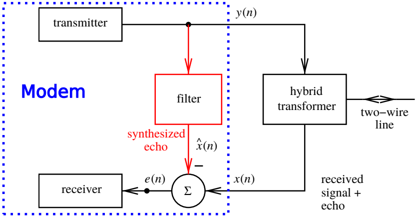


Figure 2: System identification using a training sequence $t(n)$ from an ergodic and stationary ensemble.

- Echo cancellation in full duplex data transmission.



Application examples

3) Inverse System identification

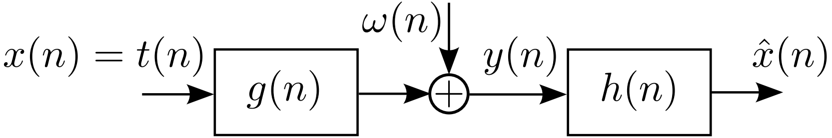


Figure 3: Inverse system identification using $x(n)$ as a training sequence.

Application examples

3) Inverse System identification

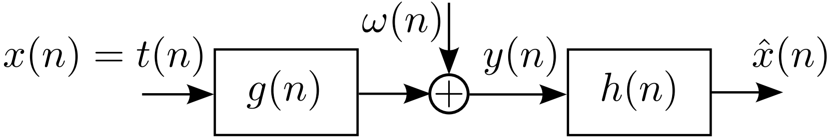
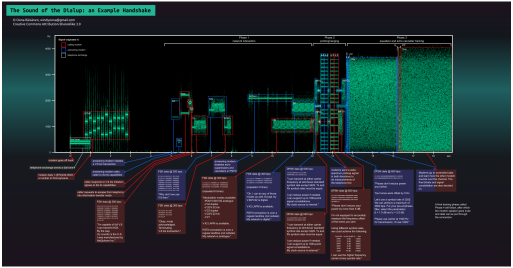


Figure 3: Inverse system identification using $x(n)$ as a training sequence.

- Channel equalisation in digital communication systems.



Optimal solution: Normal equations

- Consider the MSE $\xi(\mathbf{h}) = E\left[(e(n))^2\right]$
- The optimal filter satisfies $\nabla\xi(\mathbf{h})|_{h_{opt}} = \mathbf{0}$. Equivalently, for all $j = \ldots, -2, -1, 0, 1, 2, \ldots$

$$\begin{aligned}
\frac{\partial \xi}{\partial h(j)} &= E\left[2e(n)\frac{\partial e(n)}{\partial h(j)}\right] \\
&= E\left[2e(n)\frac{\partial\left(x(n) - \sum_{i=-\infty}^{\infty} h(i)y(n-i)\right)}{\partial h(j)}\right] \\
&= E\left[2e(n)\frac{\partial\left(-h(j)y(n-j)\right)}{\partial h(j)}\right] \\
&= -2E\left[e(n)y(n-j)\right]
\end{aligned}$$

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&= -2E\left[e(n)y(n-j)\right]
\end{aligned}
$$

- Hence, the optimal filter solves the “normal equations”

$$
E\left[e(n)y(n-j)\right] = 0, j = \ldots, -2, -1, 0, 1, 2, \ldots
$$

Optimal solution: Wiener-Hopf equations

- The error of h_{opt} is orthogonal to its observations, i.e., for all $j \in \mathbb{Z}$

$$E\left[e_{opt}(n)y(n-j)\right] = 0$$

which is known as "the principle of orthogonality".

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- Furthermore,

$$\begin{aligned} E\left[e_{opt}(n)y(n-j)\right] &= E\left[\left(x(n) - \sum_{i=-\infty}^{\infty} h_{opt}(i)y(n-i)\right)y(n-j)\right] \\ &= E\left[x(n)y(n-j)\right] - \sum_{i=-\infty}^{\infty} h_{opt}(i)E\left[y(n-i)y(n-j)\right] = 0 \end{aligned}$$

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$$= E\left[x(n)y(n-j)\right] - \sum_{i=-\infty}^{\infty} h_{opt}(i)E\left[y(n-i)y(n-j)\right] = 0$$

Result (Wiener-Hopf equations)

$$\sum_{i=-\infty}^{\infty} h_{opt}(i)r_{yy}(i-j) = r_{xy}(j)$$

The Wiener filter

- Wiener-Hopf equations can be solved indirectly, in the complex spectral domain:

$$h_{opt}(n) * r_{yy}(n) = r_{xy}(n) \leftrightarrow H_{opt}(z)P_{yy}(z) = P_{xy}(z)$$

The Wiener filter

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$$H_{opt}(z) = \frac{P_{xy}(z)}{P_{yy}(z)}$$

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- The optimal filter has an infinite impulse response (IIR), and, is non-causal, in general.

Causal Wiener filter

- We project the unconstrained solution $H_{opt}(z)$ onto the set of causal and stable IIR filters by a two step procedure:
- First, factorise $P_{yy}(z)$ into causal (right sided) $Q_{yy}(z)$, and anti-causal (left sided) parts $Q^*_{yy}(1/z^*)$, i.e., $P_{yy}(z) = \sigma_y^2 Q_{yy}(z) Q^*_{yy}(1/z^*)$.
- Select the causal (right sided) part of $P_{xy}(z)/Q^*_{yy}(1/z^*)$.

Result (Causal Wiener filter)

$$H^+_{opt}(z) = \frac{1}{\sigma_y^2 Q_{yy}(z)} \left[\frac{P_{xy}(z)}{Q^*_{yy}(1/z^*)} \right]_+$$

FIR Wiener-Hopf equations

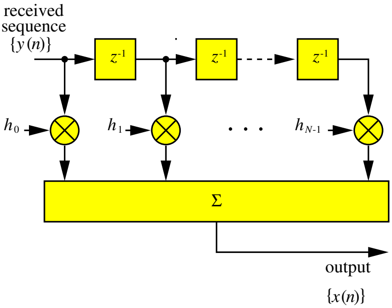

Figure 4: A finite impulse response (FIR) estimator.

- Wiener-Hopf equations for the FIR optimal filter of N taps:

Result (FIR Wiener-Hopf equations)

$$\sum_{i=0}^{N-1} h_{opt}(i) r_{yy}(i-j) = r_{xy}(j), \text{ for } j = 0, 1, ..., N-1.$$

FIR Wiener Filter

- FIR Wiener-Hopf equations in vector-matrix form.

$$\underbrace{\begin{bmatrix} r_{yy}(0) & r_{yy}(1) & \dots & r_{yy}(N-1) \\ r_{yy}(1) & r_{yy}(0) & \dots & r_{yy}(N-2) \\ \vdots & \vdots & \vdots & \vdots \\ r_{yy}(N-1) & r_{yy}(N-2) & \dots & r_{yy}(0) \end{bmatrix}}_{\triangleq \mathbf{R}_{yy}\text{ : Autocorrelation matrix of } y(n) \text{ which is Toeplitz.}} \underbrace{\begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(N-1) \end{bmatrix}}_{\triangleq \mathbf{h}_{opt}} = \underbrace{\begin{bmatrix} r_{xy}(0) \\ r_{xy}(1) \\ \vdots \\ r_{xy}(N-1) \end{bmatrix}}_{\triangleq \mathbf{r}_{xy}}$$

FIR Wiener Filter

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Result (FIR Wiener filter)

$$\mathbf{h}_{opt} = \mathbf{R}_{yy}^{-1}\mathbf{r}_{xy}.$$

MSE surface

- MSE is a quadratic function of **h**

$$\xi(\mathbf{h}) = \mathbf{h}^T \mathbf{R}_{yy}\mathbf{h} - 2\mathbf{h}^T \mathbf{r}_{xy} + E\left[(x(n))^2\right]$$

$$\nabla\xi(\mathbf{h}) = 2\mathbf{R}_{yy}\mathbf{h} - 2\mathbf{r}_{xy}$$

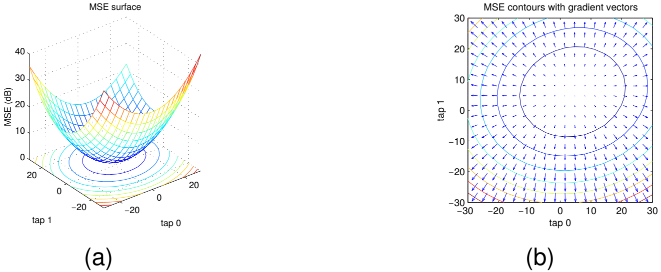


Figure 5: For a 2-tap Wiener filtering example: (a) the MSE surface, (b) gradient vectors.

Example: Wiener equaliser

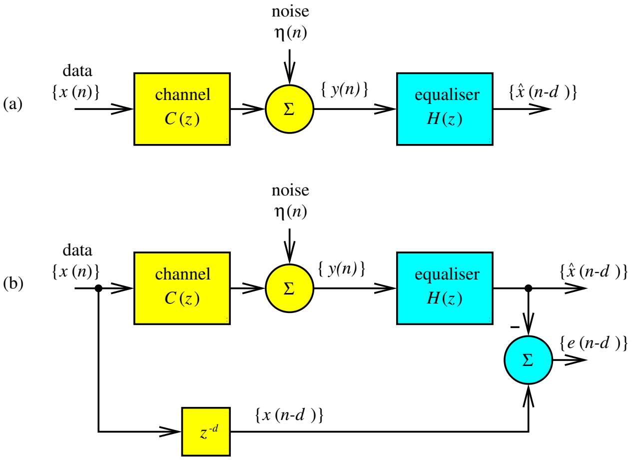


Figure 6: (a) The Wiener equaliser. (b) Alternative formulation.

Wiener equaliser

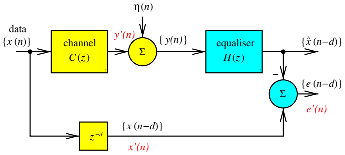


Figure 7: Channel equalisation scenario.

- For notational convenience define:

$$\begin{aligned} x'(n) &= x(n-d) \\ e'(n) &= x(n-d) - \hat{x}(n-d) \end{aligned} \quad (1)$$

- Label the output of the channel filter as $y'(n)$ where

$$y(n) = y'(n) + \eta(n)$$

Wiener equaliser

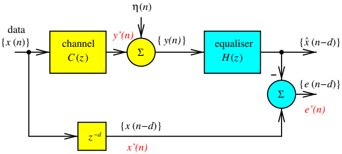


Figure 7: Channel equalisation scenario.

- Wiener filter

$$\mathbf{h}_{opt} = \mathbf{R}_{yy}{}^{-1}\mathbf{r}_{x'y} \tag{2}$$

Wiener equaliser

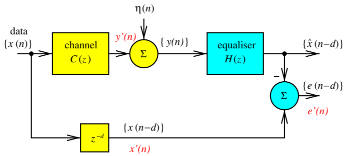


Figure 7: Channel equalisation scenario.

- Wiener filter

$$\mathbf{h}_{opt} = \mathbf{R}_{yy}{}^{-1}\mathbf{r}_{x'y} \tag{2}$$

- The (i,j)th entry in $\mathbf{R}_{yy}$ is

$$\begin{aligned} r_{yy}(j-i) &= E\left[y(j)y(i)\right] \\ &= E\left[(y'(j)+\eta(j))(y'(i)+\eta(i))\right] \\ &= r_{y'y'}(j-i)+\sigma_\eta^2\delta(j-i) \\ &\leftrightarrow P_{yy}(z) = P_{y'y'}(z)+\sigma_\eta^2 \end{aligned}$$

Wiener equaliser

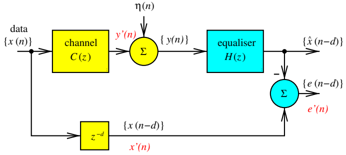


Figure 7: Channel equalisation scenario.

- Remember $y'(n) = c(n) * x(n)$

$$\leftrightarrow r_{y'y'} = c(n) * c(-n) * r_{xx}(n) \leftrightarrow P_{y'y'}(z) = C(z)C(z^{-1})P_{xx}(z)$$

- Consider a white data sequence $x(n)$, i.e.,

$$r_{xx}(n) = \sigma_x^2 \delta(n) \leftrightarrow P_{xx}(z) = \sigma_x^2.$$

- Then, the complex spectra of the autocorrelation sequence of interest is

$$P_{yy}(z) = P_{y'y'}(z) + \sigma_x^2 = C(z)C(z^{-1})\sigma_x^2 + \sigma_\eta^2$$

Wiener equaliser

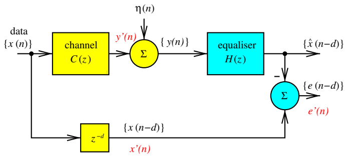


Figure 7: Channel equalisation scenario.

- Wiener filter

$$\mathbf{h}_{opt} = \mathbf{R}_{yy}{}^{-1}\mathbf{r}_{x'y} \quad (3)$$

Wiener equaliser

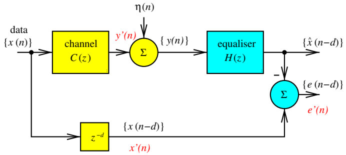


Figure 7: Channel equalisation scenario.

- Wiener filter

$$\mathbf{h}_{opt} = \mathbf{R}_{yy}{}^{-1}\mathbf{r}_{x'y} \qquad (3)$$

- The (j)th entry in $\mathbf{r}_{x'y}$ is

$$\begin{aligned} r_{x'y}(j) &= E\left[x'(n)y(n-j)\right] \\ &= E\left[x(n-d)(y'(n-j)+\eta(n-j))\right] \\ &= r_{xy'}(j-d) \\ &\leftrightarrow P_{x'y}(z) = P_{xy'}(z)z^{-d} \end{aligned} \qquad (4)$$

Wiener equaliser

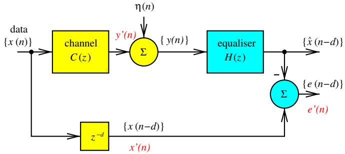


Figure 7: Channel equalisation scenario.

- Remember $y'(n) = c(n) * x(n)$

$$\leftrightarrow r_{xy'} = c(-n) * r_{xx}(n) \leftrightarrow P_{xy'}(z) = C(z^{-1})P_{xx}(z)$$

- Then, the complex spectra of the cross correlation sequence of interest is

$$P_{x'y}(z) = P_{xy'}(z)z^{-d} = \sigma_x^2 C(z^{-1})z^{-d}$$

Wiener equaliser

- Suppose that $\mathbf{c} = [c(0) = 0.5, c(1) = 1]^T \leftrightarrow C(z) = (0.5 + z^{-1})$
- Then,

$$P_{yy}(z) = C(z)C(z^{-1})\sigma_x^2 + \sigma_\eta^2 = (0.5 + z^{-1})(0.5 + z)\sigma_x^2 + \sigma_\eta^2$$
$$P_{x'y}(z) = \sigma_x^2 C(z^{-1})z^{-d} = (0.5z^{-d} + z^{-d+1})\sigma_x^2$$

- Suppose that $d = 1$, $\sigma_x^2 = 1$, and, $\sigma_\eta^2 = 0.1$

$$r_{yy}(0) = 1.35,\ r_{yy}(1) = 0.5, \text{ and } r_{yy}(2) = 0$$
$$r_{x'y}(0) = 1,\ r_{x'y}(1) = 0.5, \text{ and } r_{x'y}(2) = 0$$

- The Wiener filter is obtained as

$$\mathbf{h}_{opt} = \left(\begin{bmatrix} 1.35 & 0.5 & 0 \\ 0.5 & 1.35 & 0.5 \\ 0 & 0.5 & 1.35 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.69 \\ 0.13 \\ -0.05 \end{bmatrix}$$

- The MSE is found as $\xi(\mathbf{h}_{opt}) = \sigma_x^2 - \mathbf{h}_{opt}^T \mathbf{r}_{x'y} = 0.24$.

Adaptive filtering - Introduction

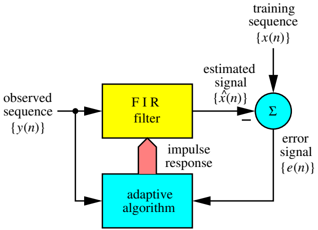


Figure 8: Adaptive filtering configuration.

- For notational convenience, define

$$\mathbf{y}(n) \triangleq [y(n), y(n-1), \ldots, y(n-N+1)]^T \quad , \quad \mathbf{h}(n) \triangleq [h_0, h_1, \ldots, h_{N-1}]^T$$

- The output of the adaptive filter is

$$\hat{x}(n) = \mathbf{h}^T(n)\mathbf{y}(n)$$

- Optimum solution

$$\mathbf{h}_{opt} = \mathbf{R}_{yy}^{-1}\mathbf{r}_{xy}$$

Recursive least squares

- Minimise cost function

$$\xi(n) = \sum_{k=0}^{n} (x(k) - \hat{x}(k))^2 \quad (5)$$

- Solution

$$\mathbf{R}_{yy}(n)\mathbf{h}(n) = \mathbf{r}_{xy}(n)$$

- LS "autocorrelation" matrix

$$\mathbf{R}_{yy}(n) = \sum_{k=0}^{n} \mathbf{y}(k)\mathbf{y}^T(k)$$

- LS "cross-correlation" vector

$$\mathbf{r}_{xy}(n) = \sum_{k=0}^{n} \mathbf{y}(k)x(k)$$

Recursive least squares

- Recursive relationships

$$\mathbf{R}_{yy}(n) = \mathbf{R}_{yy}(n-1) + \mathbf{y}(n)\mathbf{y}^T(n)$$
$$\mathbf{r}_{xy}(n) = \mathbf{r}_{xy}(n-1) + \mathbf{y}(n)x(n)$$

- Substitute for $\mathbf{r}_{xy}$

$$\mathbf{R}_{yy}(n)\mathbf{h}(n) = \mathbf{R}_{yy}(n-1)\mathbf{h}(n-1) + \mathbf{y}(n)x(n)$$

- Replace $\mathbf{R}_{yy}(n-1)$

$$\mathbf{R}_{yy}(n)\mathbf{h}(n) = \left(\mathbf{R}_{yy}(n) - \mathbf{y}(n)\mathbf{y}^T(n)\right)\mathbf{h}(n-1) + \mathbf{y}(n)x(n)$$

- Multiple both sides by $\mathbf{R}_{yy}^{-1}(n)$

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mathbf{R}_{yy}^{-1}(n)\mathbf{y}(n)e(n)$$
$$e(n) = x(n) - \mathbf{h}^T(n-1)\mathbf{y}(n)$$

Recursive least squares

- Recursive relationships

$$\mathbf{R}_{yy}(n) = \mathbf{R}_{yy}(n-1) + \mathbf{y}(n)\mathbf{y}^T(n)$$

- Apply Sherman-Morrison identity

$$\mathbf{R}_{yy}^{-1}(n) = \mathbf{R}_{yy}^{-1}(n-1) - \frac{\mathbf{R}_{yy}^{-1}(n-1)\mathbf{y}(n)\mathbf{y}^T(n)\mathbf{R}_{yy}^{-1}(n-1)}{1 + \mathbf{y}^T(n)\mathbf{R}_{yy}^{-1}(n-1)\mathbf{y}(n)}$$

Summary

Recursive least squares (RLS) algorithm:

$\mathbf{R}_{yy}(0) = \frac{1}{\delta}\mathbf{I}_N$ with small positive δ ▷ Initialisation 1

$\mathbf{h}(0) = \mathbf{0}$ ▷ Initialisation 2

for $n = 1, 2, 3, \ldots$ **do** ▷ Iterations

$\quad \hat{x}(n) = \mathbf{h}^T(n-1)\mathbf{y}(n)$ ▷ Estimate $x(n)$

$\quad e(n) = x(n) - \hat{x}(n)$ ▷ Find the error

$$\mathbf{R}_{yy}^{-1}(n) = \frac{1}{\alpha}\left(\mathbf{R}_{yy}^{-1}(n-1) - \frac{\mathbf{R}_{yy}^{-1}(n-1)\mathbf{y}(n)\mathbf{y}^T(n)\mathbf{R}_{yy}^{-1}(n-1)}{\alpha + \mathbf{y}^T(n)\mathbf{R}_{yy}^{-1}(n-1)\mathbf{y}(n)}\right)$$

▷ Update the inverse of the autocorrelation matrix

$\quad \mathbf{h}(n) = \mathbf{h}(n-1) + \mathbf{R}_{yy}^{-1}(n)\mathbf{y}(n)e(n)$ ▷ Update the filter coefficients

end for

Stochastic gradient algorithms

- MSE contour - 2-tap example:

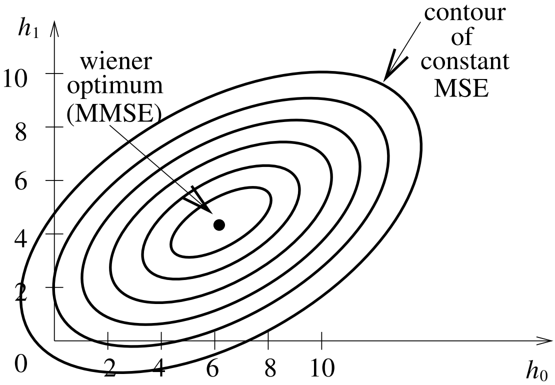
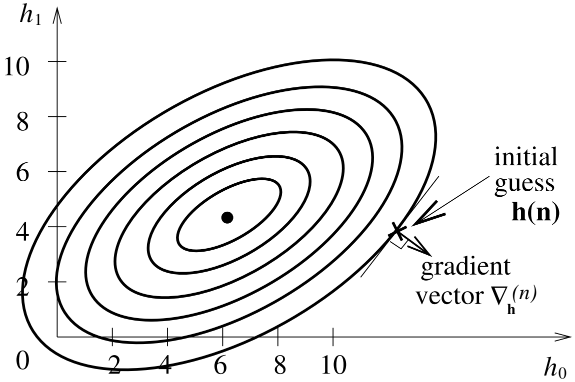


Figure 9: Method of steepest descent.

Steepest descent

- MSE contour - 2-tap example:

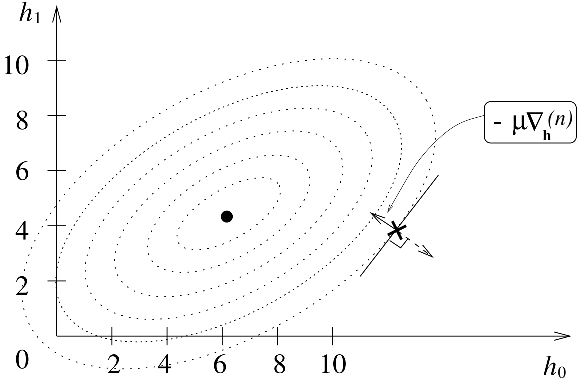


- The gradient vector

$$\nabla_{\mathbf{h}}(n) = \left[\frac{\partial \xi}{\partial h(0)}, \frac{\partial \xi}{\partial h(1)}, \ldots, \frac{\partial \xi}{\partial h(N-1)} \right]^T \Bigg|_{\mathbf{h}=\mathbf{h}(n)} = 2\mathbf{R}_{yy}\mathbf{h}(n) - 2\mathbf{r}_{xy}$$

Steepest descent

- MSE contour - 2-tap example:



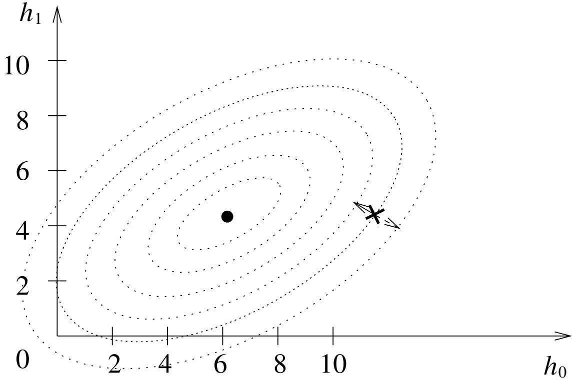
- Update initial guess in the direction of steepest descent:

$$\mathbf{h}(n+1) = \mathbf{h}(n) - \mu \nabla_{\mathbf{h}}(n)$$

- Step-size μ.

Steepest descent

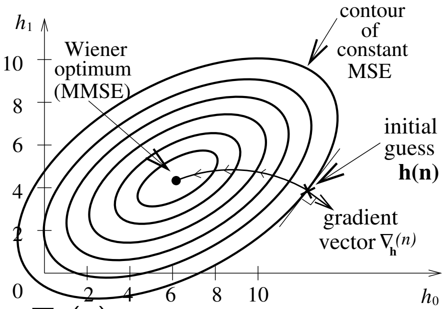
- MSE contour - 2-tap example:



- Gradient at new guess.

Convergence of steepest descent

- MSE contour - 2-tap example:



$$\mathbf{h}(n+1) = \mathbf{h}(n) - \mu \nabla_{\mathbf{h}}(n)$$

$$\nabla_{\mathbf{h}}(n) = \left[\frac{\partial \xi}{\partial h(0)}, \frac{\partial \xi}{\partial h(1)}, \ldots, \frac{\partial \xi}{\partial h(N-1)} \right]^T \Bigg|_{\mathbf{h}=\mathbf{h}(n)} = 2\mathbf{R}_{yy}\mathbf{h}(n) - 2\mathbf{r}_{xy}$$

$$0 < \mu < \frac{1}{\lambda_{max}} \quad (6)$$

Stochastic gradient algorithms

- A time recursion:

$$\mathbf{h}(n+1) = \mathbf{h}(n) - \mu \hat{\nabla}_{\mathbf{h}}(n)$$

- The exact gradient:

$$\begin{aligned}\nabla_{\mathbf{h}}(n) &= -2E\left[\mathbf{y}(n)(x(n) - \mathbf{y}(n)^T\mathbf{h}(n))\right] \\ &= -2E\left[\mathbf{y}(n)e(n)\right]\end{aligned}$$

- A simple estimate of the gradient

$$\hat{\nabla}_{\mathbf{h}}(n) = -2\mathbf{y}(n+1)e(n+1)$$

- The error

$$e(n+1) = x(n+1) - \mathbf{h}(n)^T\mathbf{y}(n+1) \tag{7}$$

The Least-mean-squares (LMS) algorithm:

$\mathbf{h}(0) = \mathbf{0}$ ▷ Initialisation

for $n = 1, 2, 3, \ldots$ **do** ▷ Iterations

  $\hat{x}(n) = \mathbf{h}^T(n-1)\mathbf{y}(n)$ ▷ Estimate $x(n)$

  $e(n) = x(n) - \hat{x}(n)$ ▷ Find the error

  $\mathbf{h}(n) = \mathbf{h}(n-1) + 2\mu\mathbf{y}(n)e(n)$ ▷ Update the filter coefficients

end for

LMS block diagram

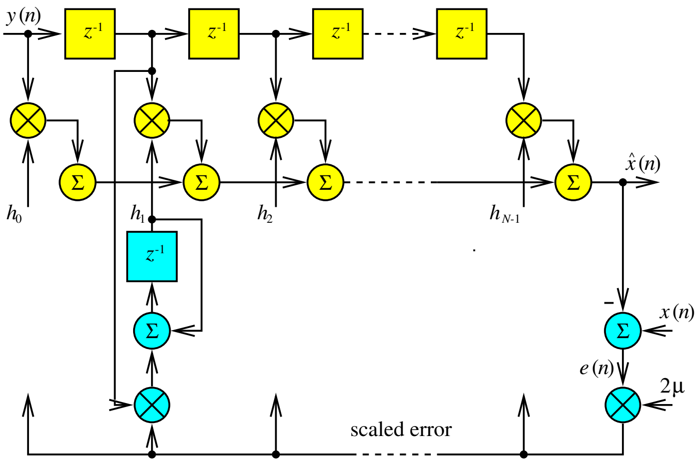
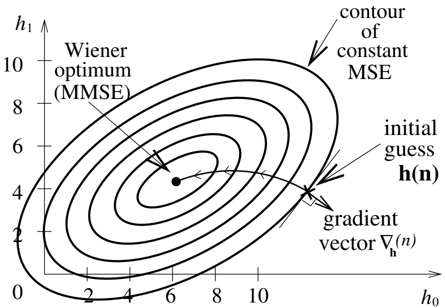


Figure 10: Least mean-square adaptive filtering.

Convergence of the LMS

- MSE contour - 2-tap example:



- Eigenvalues of $\mathbf{R}_{yy}$ (in this example, λ_0 and λ_1).
- The largest time constant $\tau_{max} > \frac{\lambda_{max}}{2\lambda_{min}}$
- Eigenvalue ratio (EVR) is $\frac{\lambda_{max}}{\lambda_{min}}$
- Practical range for step-size $0 < \mu < \frac{1}{3N\sigma_y^2}$

Eigenvalue ratio (EVR)

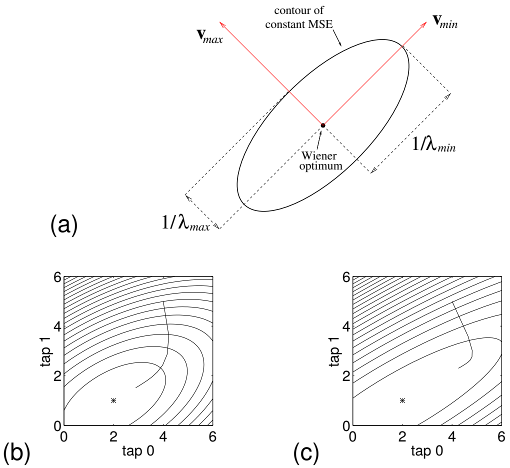


Figure 11: Eigenvectors, eigenvalues and convergence: (a) the relationship between eigenvectors, eigenvalues and the contours of constant MSE; (b) steepest descent for EVR of 2; (c) EVR of 4.

Comparison of RLS and LMS

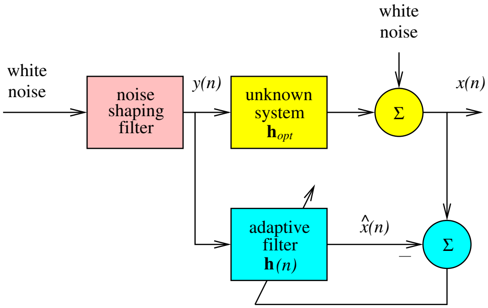


Figure 12: Adaptive system identification configuration.

- Error vector norm

$$\rho(n) = E\left[(\mathbf{h}(n) - \mathbf{h}_{opt})^T(\mathbf{h}(n) - \mathbf{h}_{opt})\right]$$

Comparison: Performance

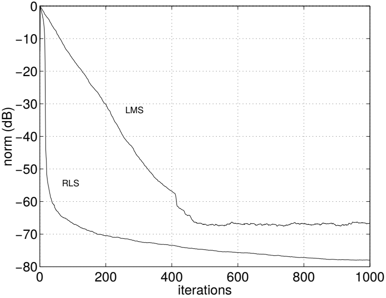


Figure 13: Covergence plots for $N = 16$ taps adaptive filtering in the system identification configuration: EVR = 1 (i.e., the impulse response of the noise shaping filter is $\delta(n)$).

Comparison: Performance

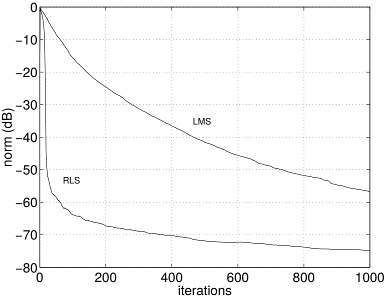


Figure 14: Covergence plots for $N = 16$ taps adaptive filtering in the system identification configuration: EVR = 11.

Comparison: Performance

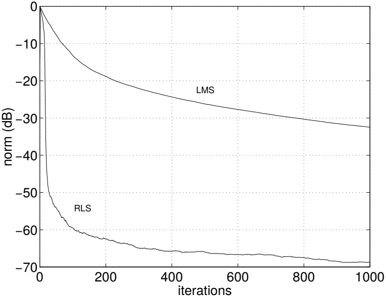


Figure 15: Covergence plots for $N = 16$ taps adaptive filtering in the system identification configuration: EVR (and, correspondingly the spectral coloration of the input signal) progressively increases to 68.

Comparison: Complexity

Table: Complexity comparison of N-point FIR filter algorithms.

Algorithm class	*Implementation*	*Computational load*		
		multiplications	*adds/subtractions*	*divisions*
RLS	fast Kalman	$10N+1$	$9N+1$	2
SG	LMS	$2N$	$2N$	–
	BLMS (via FFT)	$10\log(N)+8$	$15\log(N)+30$	–

Applications

- Adaptive filtering algorithms can be used in all application areas of optimal filtering.
- Some examples:
  - Adaptive line enhancement
  - Adaptive tone suppression
  - Echo cancellation
  - Channel equalisation

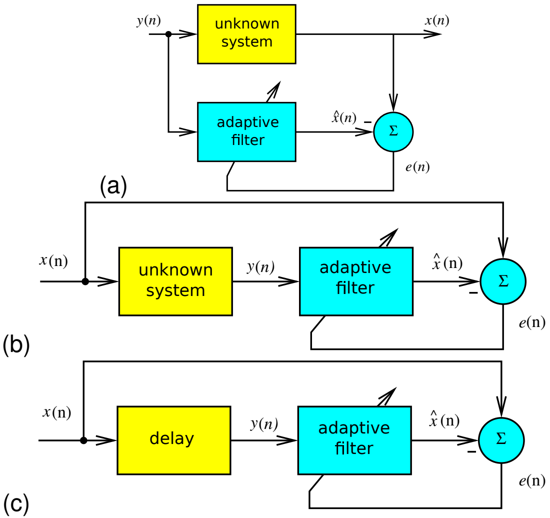


Figure 16: Adaptive filtering configurations: (a) direct system modelling; (b) inverse system modelling; (c) linear prediction.

Adaptive line enhancement

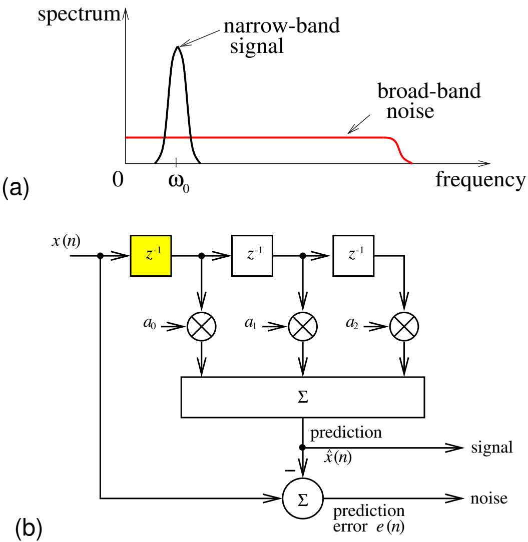
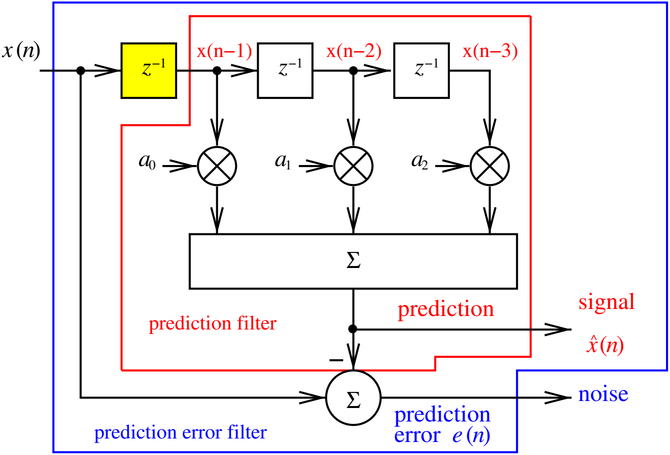


Figure 17: Adaptive line enhacement: (a) signal spectrum; (b) system

Adaptive predictor



- Prediction filter: $a_0 + a_1 z^{-1} + a_2 z^{-2}$
- Prediction error filter: $1 - a_0 z^{-1} - a_1 z^{z-2} - a_2 z^{-3}$

Adaptive tone suppression

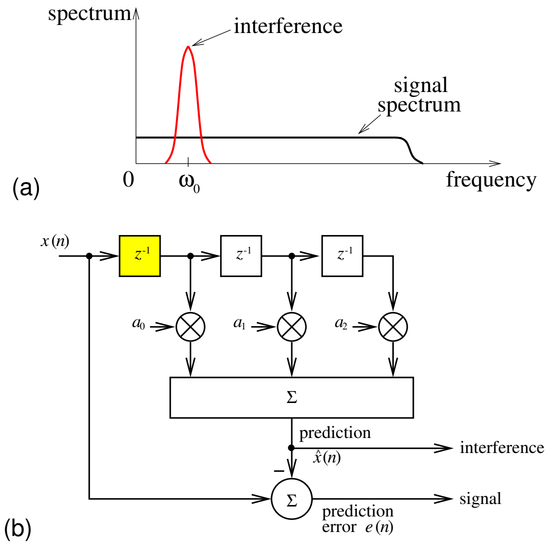


Figure 18: Adaptive tone suppression: (a) signal spectrum; (b) system

Adaptive noise whitening

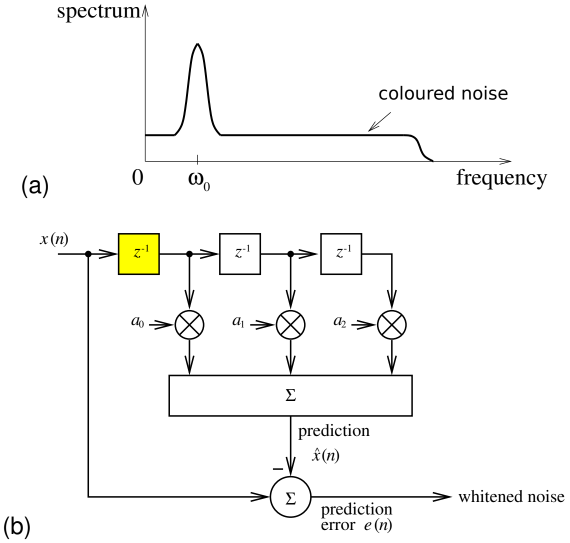
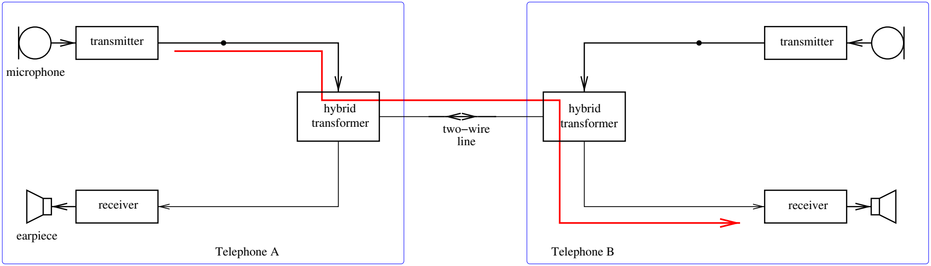


Figure 19: Adaptive noise whitening: (a) input spectrum; (b) system

Echo cancellation

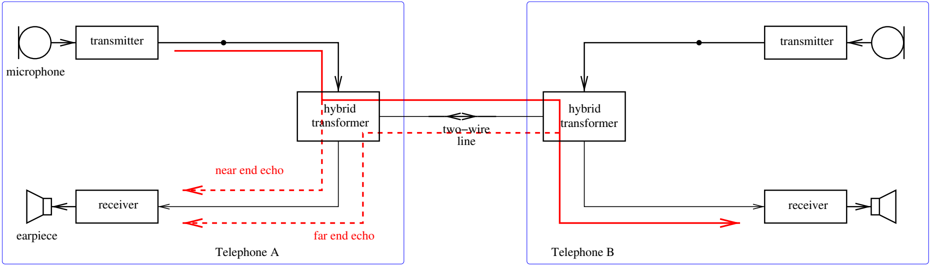
- A typical telephone connection



- Hybrid transformers to route signal paths.

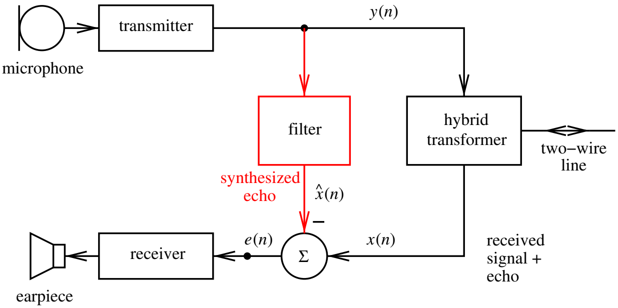
Echo cancellation (contd)

- Echo paths in a telephone system



- Near and far echo paths.

Echo cancellation (contd)



- Fixed filter?

Echo cancellation (contd)

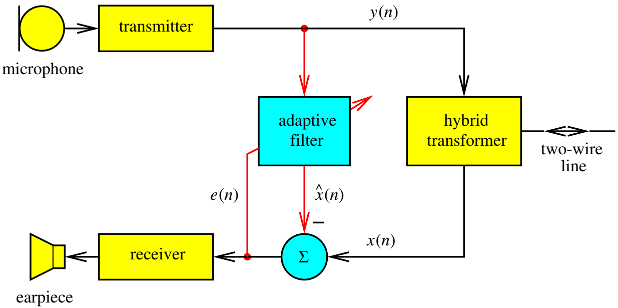


Figure 20: Application of adaptive echo cancellation in a telephone handset.

Channel equalisation

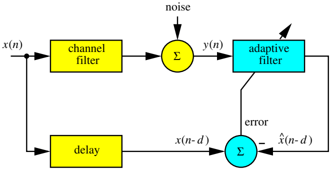


Figure 21: Adaptive equaliser system configuration.

- Simple channel

$$y(n) = \pm h_0 + \text{noise}$$

- Decision circuit

$$\text{if } y(n) \geq 0 \text{ then } x(n) = +1 \text{ else} x(n) = -1$$

- Channel with intersymbol interference (ISI)

$$y(n) = \sum_{i=0}^{2} h_i x(n-i) + \text{noise}$$

Channel equalisation

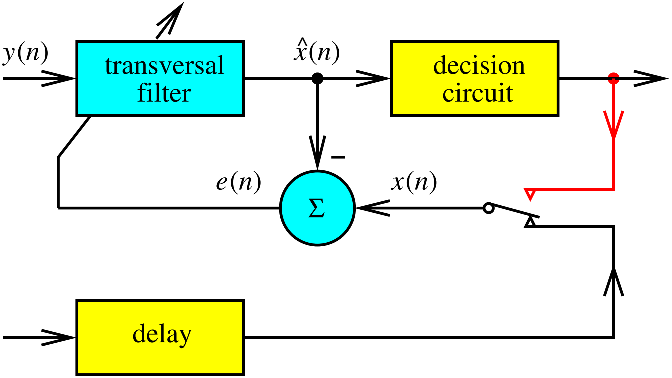


Figure 22: Decision directed equaliser.

Optimal signal detection - Introduction

Optimal signal detection - Introduction

- Example: Detection of gravitational waves

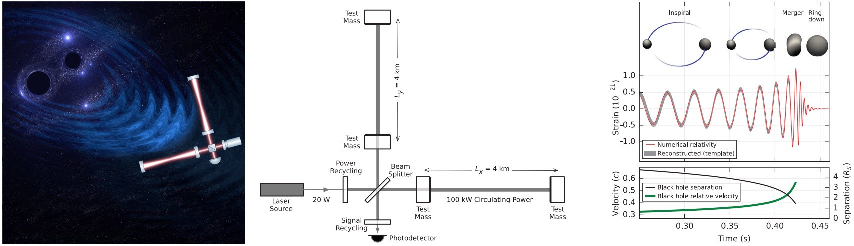


Figure 23: Gravitational waves from a binary black hole merger (left, Uni. of Birmingham, Gravitational Wave Group), LIGO block diagram (middle), expected signal (right) Abbot et. al.,“Observation of gravitational waves from a binary black hole merger”, Phys. Rev. Let., Feb. 2016..

Optimal signal detection

- Signal detection as 2-ary (binary) hypothesis testing:

$$\begin{aligned} H_0 &: y(n) = \eta(n) \\ H_1 &: y(n) = x(n) + \eta(n) \end{aligned} \quad (8)$$

- In a sense, decide which of the two possible ensembles $y(n)$ is generated from.
- Finite length signals, i.e.,

$$n = 0, 1, 2, ..., N - 1$$

- Vector notation

$$\begin{aligned} H_0 &: \mathbf{y} = \boldsymbol{\eta} \\ H_1 &: \mathbf{y} = \mathbf{x} + \boldsymbol{\eta} \end{aligned}$$

Optimal signal detection

- Example (radar): In active sensing, $x(t)$ is the probing waveform subject to design.

$$y(n) = a_0 x(n - n_0) + a_1 x(n - n_1) + \cdots + \eta(n) \qquad (9)$$

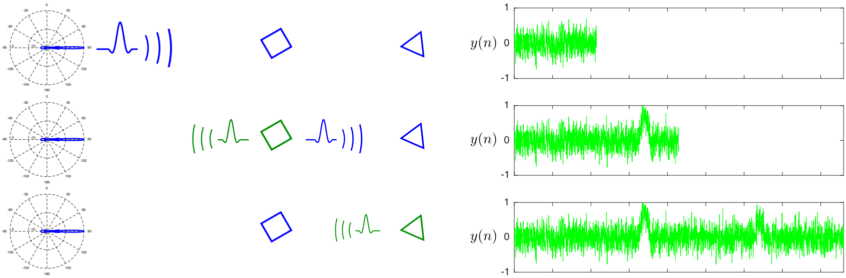


Figure 24: Probing waveform $x(n)$ and returns from the surveillance region constituting $y(n)$.

Optimal signal detection

- Example (radar): In active sensing, $x(t)$ is the probing waveform subject to design.

$$y(n) = a_0 x(n - n_0) + a_1 x(n - n_1) + \cdots + \eta(n) \quad (9)$$

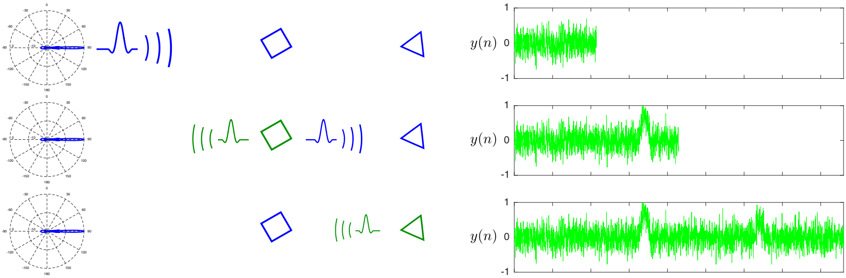


Figure 24: Probing waveform $x(n)$ and returns from the surveillance region constituting $y(n)$.

- A similar problem also arises in digital communications.

Bayesian hypothesis testing

- Consider a random variable **H** with $H = \mathbf{H}(\zeta)$ and $H \in \{H_0, H_1\}$.
- The measurement $\bar{y} = \mathbf{y}(\zeta)$ is a realisation of **y**.
- The measurement model specifies the likelihood $p_{\mathbf{Y}|H}(\bar{y}|H)$
- Find the probabilities of $H = H_1$ and $H = H_0$ based on the measurement vector $\bar{y}$.
- Decide on the hypothesis with the maximum a posteriori probability:

Bayesian hypothesis testing

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- The measurement model specifies the likelihood $p_{\mathbf{Y}|H}(\bar{y}|H)$
- Find the probabilities of $H = H_1$ and $H = H_0$ based on the measurement vector $\bar{y}$.
- Decide on the hypothesis with the maximum a posteriori probability:
- Find the *maximum a-posteriori* (MAP) estimate of H.

$$\hat{H} = \arg \max_{H \in \{H_0, H_1\}} P_{\mathbf{H}|\mathbf{y}}(H|\bar{y}) \tag{10}$$

where the a posterior probability of **H** is given by

$$P_{\mathbf{H}|\mathbf{y}}(H_i|\bar{y}) = \frac{p_{\mathbf{Y}|H}(\bar{y}|H_i) P_{\mathbf{H}}(H_i)}{p_{\mathbf{Y}|H}(\bar{y}|H_0) P_{\mathbf{H}}(H_0) + p_{\mathbf{Y}|H}(\bar{y}|H_1) P_{\mathbf{H}}(H_1)} \tag{11}$$

for $i = 0, 1$.

Bayesian hypothesis testing: The likelihood ratio test

- MAP decision rule:

$$\hat{H} = \arg \max_{H \in \{H_0, H_1\}} P(H|\bar{y})$$

- MAP decision as a likelihood ratio test:

$$p(H_1|\bar{y}) \underset{\hat{H}=H_0}{\overset{\hat{H}=H_1}{\gtrless}} p(H_0|\bar{y})$$

$$p(\bar{y}|H_1)P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} p(\bar{y}|H_0)P(H_0)$$

$$\frac{p(\bar{y}|H_1)}{p(\bar{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

Bayesian hypothesis testing - AWGN Example

- Example: Detection of deterministic signals in additive white Gaussian noise:

$$H_0 : \mathbf{y} = \boldsymbol{\eta}$$
$$H_1 : \mathbf{y} = \mathbf{x} + \boldsymbol{\eta}$$

where $\mathbf{x}$ is a known vector, $\boldsymbol{\eta} \sim \mathcal{N}(.; \mathbf{0}, \sigma^2 \mathbf{I})$.

- The likelihoods are specified by the noise distribution:

$$\frac{p(\bar{y}|H_1)}{p(\bar{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$
$$\frac{\mathcal{N}(\bar{y}; \mathbf{x}, \sigma^2 \mathbf{I})}{\mathcal{N}(\bar{y}; \mathbf{0}, \sigma^2 \mathbf{I})} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

Detection of deterministic signals - AWGN (contd)

- The numerator and denominator of the likelihood ratio are

$$p(\bar{y}|H_1) = \mathcal{N}(\bar{y} - \mathbf{x}; \mathbf{0}, \sigma^2 \mathbf{I})$$
$$= \frac{1}{(2\pi\sigma^2)^{N/2}} \prod_{n=0}^{N-1} \exp\left\{ -\frac{(\bar{y}(n) - x(n))^2}{2\sigma^2} \right\}$$
$$= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \left(\sum_{n=0}^{N-1} (\bar{y}(n) - x(n))^2 \right) \right\} \quad (12)$$

- Similarly

$$p(\bar{y}|H_0) = \mathcal{N}(\bar{y}; \mathbf{0}, \sigma^2 \mathbf{I})$$
$$= \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{ -\frac{1}{2\sigma^2} \left(\sum_{n=0}^{N-1} (\bar{y}(n))^2 \right) \right\} \quad (13)$$

- Therefore

$$\frac{p(\bar{y}|H_1)}{p(\bar{y}|H_0)} = \exp\left\{ \frac{1}{\sigma^2} \left(\sum_{n=0}^{N-1} (\bar{y}(n)x(n) - \frac{1}{2} x(n)^2) \right) \right\} \quad (14)$$

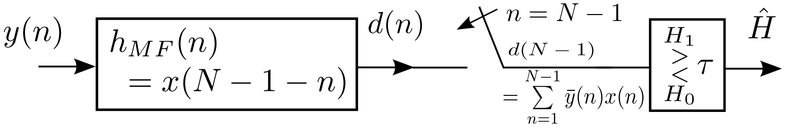
Detection of deterministic signals - AWGN (contd)

- Take the logarithm of both sides of the likelihood ratio test

$$\log \exp \left\{ \frac{1}{\sigma^2} \left(\sum_{n=0}^{N-1} (\bar{y}(n)x(n) - \frac{1}{2}x(n)^2) \right) \right\} \underset{H_0}{\overset{H_1}{\gtrless}} \log \frac{P(H_0)}{P(H_1)}$$

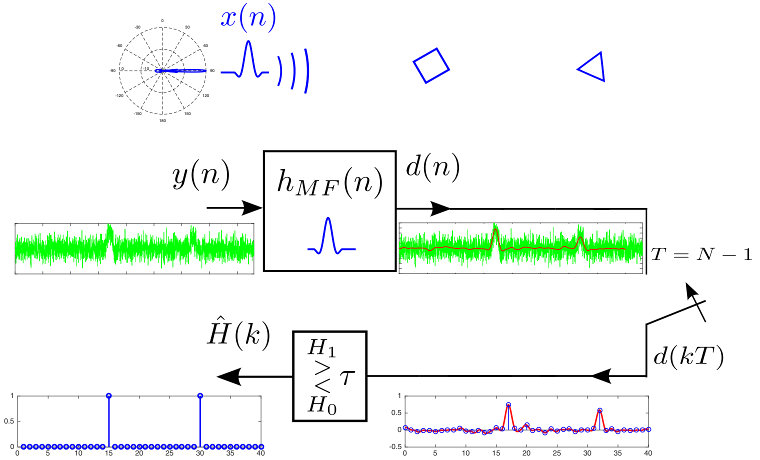
- Now, we have a linear statistical test

$$\sum_{n=0}^{N-1} \bar{y}(n)x(n) \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\sigma^2 \log \frac{P(H_0)}{P(H_1)} + \frac{1}{2} \sum_{n=0}^{N-1} x(n)^2}_{\triangleq \tau\text{:Decision threshold}}$$



Detection of deterministic signals - AWGN (contd)

- Matched filtering for optimal detection:

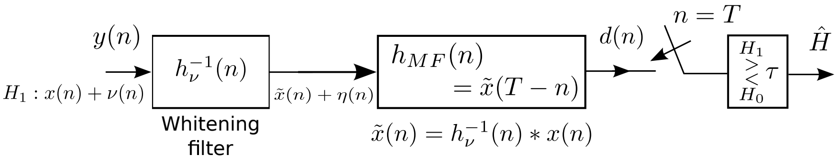


Detection of deterministic signals under coloured noise

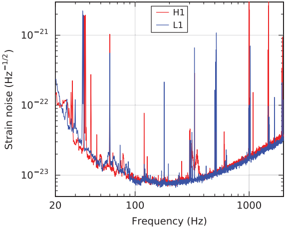
- For the case, the noise sequence $\rho(n)$ has an auto-correlation function $r_\nu[l]$ different than $r_\nu[l] = \sigma_\eta^2 \times \delta[l]$, and,

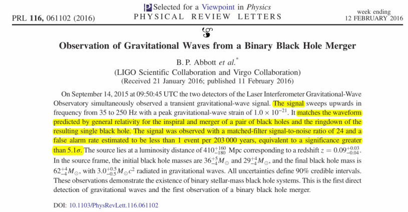
$$H_0 : \mathbf{y} = \boldsymbol{\nu}$$
$$H_1 : \mathbf{y} = \mathbf{x} + \boldsymbol{\nu}$$

$$\boldsymbol{\nu} \sim \mathcal{N}\left(.; \mathbf{0}, \mathbf{C}_\nu = \begin{bmatrix} r_\nu(0) & r_\nu(-1) & \dots & r_\nu(-N+1) \\ r_\nu(1) & r_\nu(0) & \dots & r_\nu(-N+2) \\ \vdots & \vdots & \vdots & \vdots \\ r_\nu(N-1) & r_\nu(N-2) & \dots & r_\nu(0) \end{bmatrix}\right)$$



Detection of deterministic signals - coloured noise ex.





Selected for a Viewpoint in *Physics*

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Observation of Gravitational Waves from a Binary Black Hole Merger

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On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21}. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. The signal was observed with a matched-filter signal-to-noise ratio of 24 and a false alarm rate estimated to be less than 1 event per 203 000 years, equivalent to a significance greater than 5.1σ. The source lies at a luminosity distance of 410^{+160}_{-180} Mpc corresponding to a redshift $z = 0.09^{+0.03}_{-0.04}$. In the source frame, the initial black hole masses are $36^{+5}_{-4} M_{\odot}$ and $29^{+4}_{-4} M_{\odot}$, and the final black hole mass is $62^{+4}_{-4} M_{\odot}$, with $3.0^{+0.5}_{-0.5} M_{\odot} c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.

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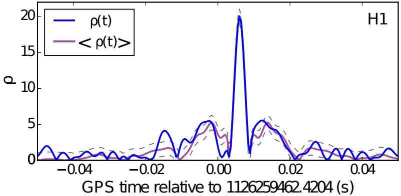

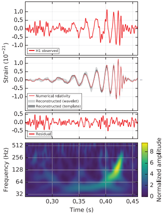

(upper left) Noise (amplitude) spectral density. (upper right) Abstract, Abbot et. al., Phys. Rev. Let., Feb. 2016.. (lower left) Matched filter outputs: Best MF (blue) and the expected MF (purple). (lower right) Measurement, reconstructed and noise signals around the detection.

Summary

- Optimal filtering: Problem statement
- General solution via Wiener-Hopf equations
- FIR Wiener filter
- Adaptive filtering as an online optimal filtering strategy
- Recursive least-squares (RLS) algorithm
- Least mean-square (LMS) algorithm
- Application examples
- Optimal signal detection via matched filtering

Further reading

- C. Therrien, *Discrete Random Signals and Statistical Signal Processing*, Prentice-Hall, 1992.
- S. Haykin, *Adaptive Filter Theory,* 5th ed., Prentice-Hall, 2013.
- B. Mulgrew, P. Grant, J. Thompson, *Digital Signal Processing: Concepts and Applications*, 2nd ed., Palgrave Macmillan, 2003.
- D. Manolakis, V. Ingle, S. Kogon, *Statistical and Adaptive Signal Processing*, McGraw Hill, 2000.