Convex Optimization

Fundamentals and Applications in Statistical Signal Processing

João Mota

EURASIP/UDRC Summer School 2020

Heriot-Watt University

Optimization Problems

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega \end{aligned}$$

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- $x \in \mathbb{R}^n$: optimization variable
- $f : \mathbb{R}^n \to \mathbb{R}$: cost function (or objective)
- $\Omega \subset \mathbb{R}^n$: constraint set

Example: Polynomial Fitting

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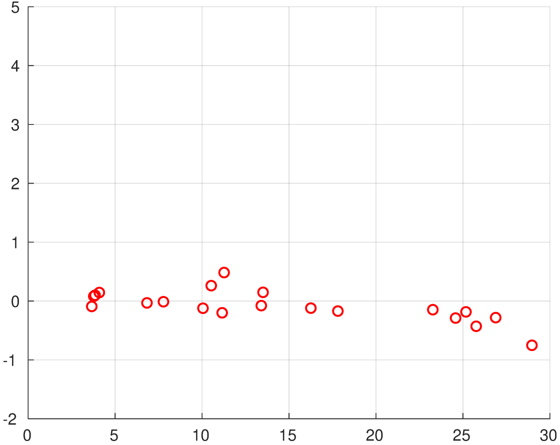

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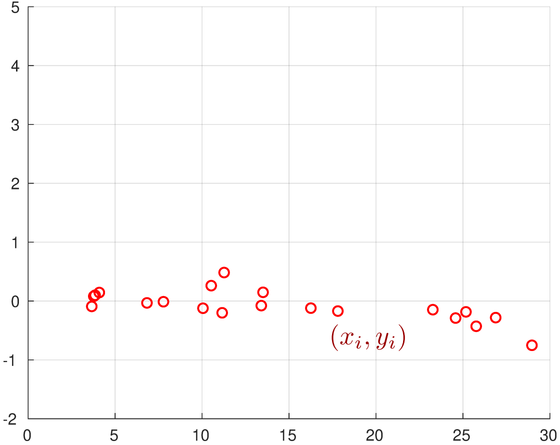

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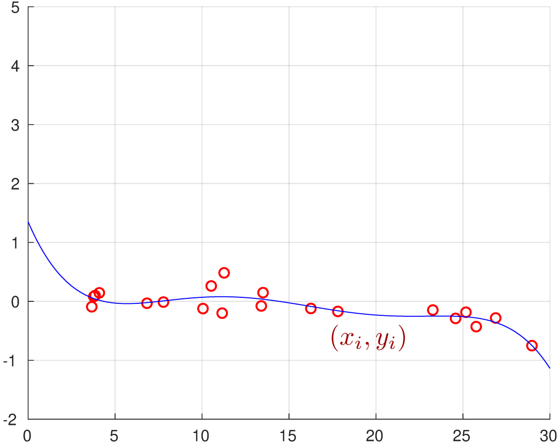

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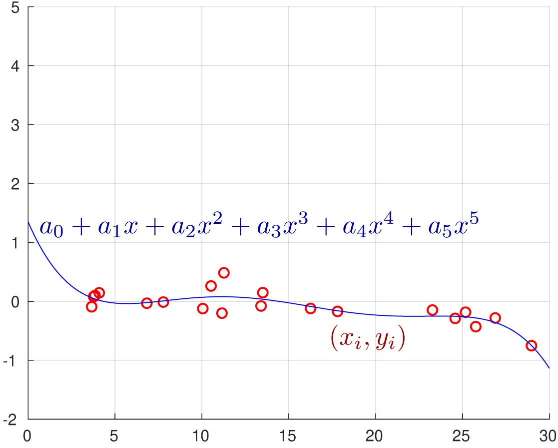

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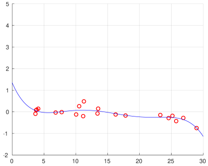

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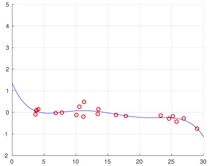

Polynomial of order $k = 5$:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

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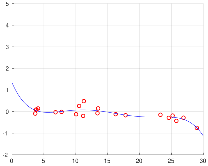

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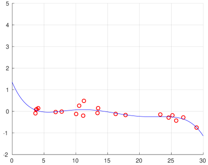

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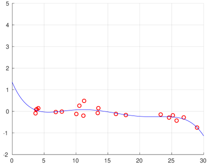

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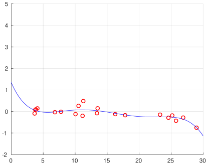

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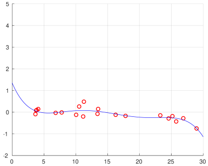

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$$\underset{a \in \mathbb{R}^5}{\text{minimize}} \quad f(a) = \left\| y - Xa \right\|_2^2 \qquad \nabla f(a^\star) = 0 \iff X^\top X a^\star = X^\top y$$

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What if there are outliers?

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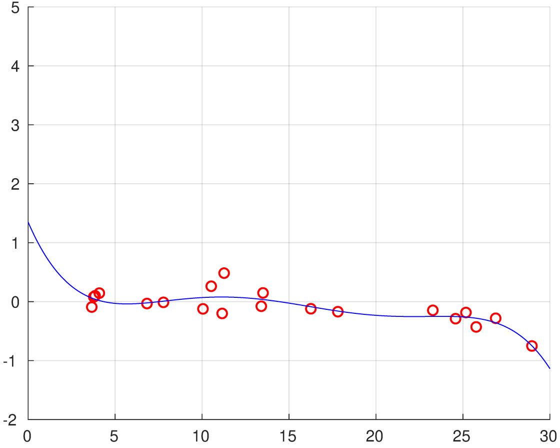

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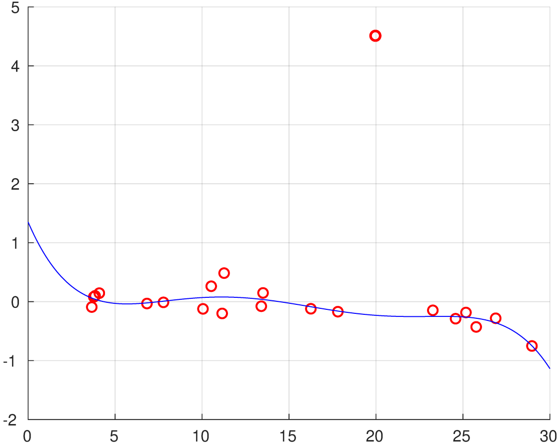

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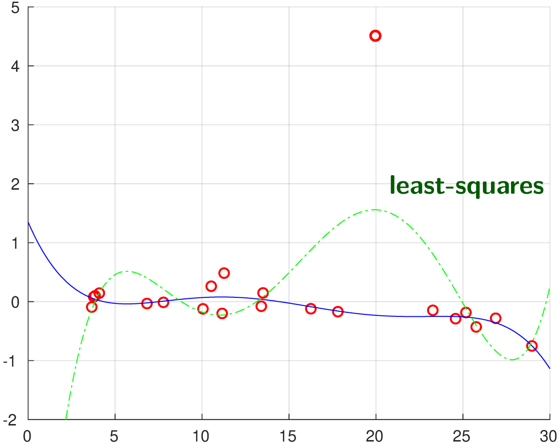

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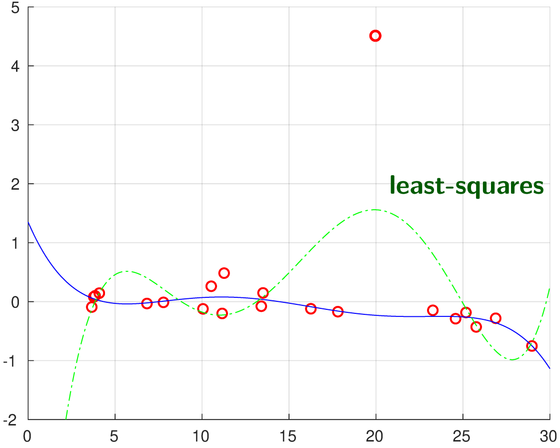

least-squares $\min_{a \in \mathbb{R}^5} \|y - Xa\|_2^2$

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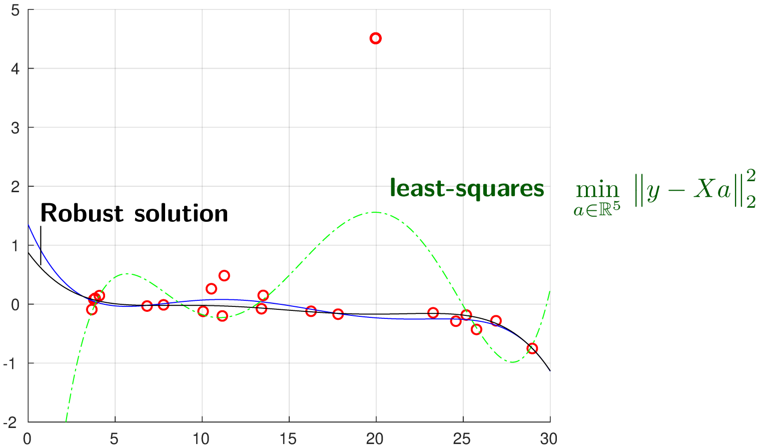

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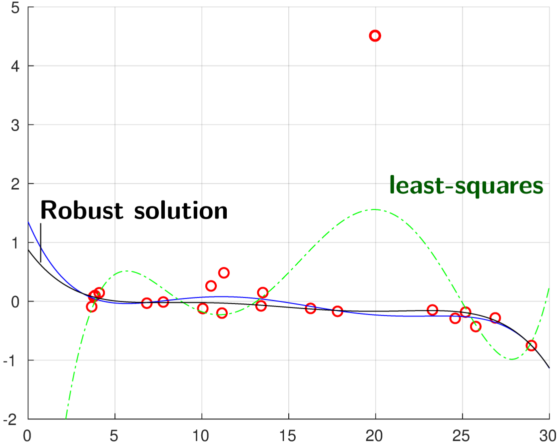

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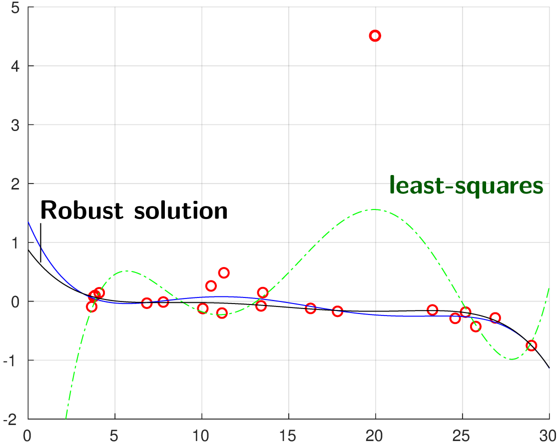

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no closed-form

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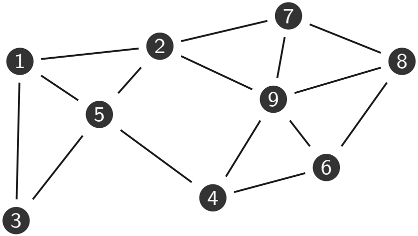

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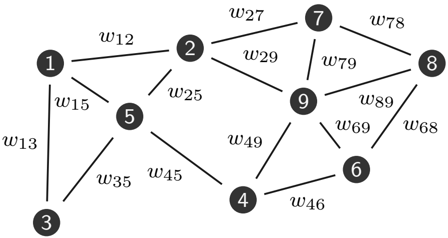

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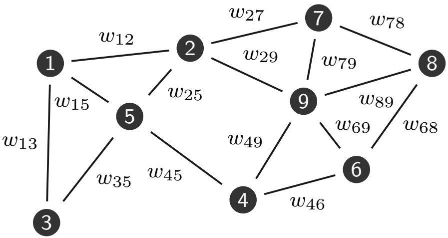

Cut: set of edges whose removal splits the graph into two

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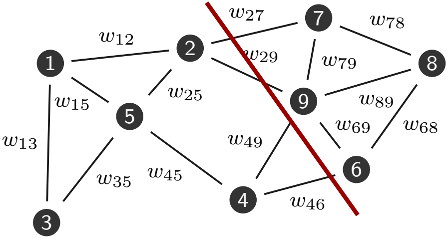

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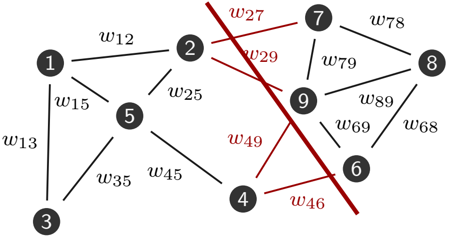

value of cut: $w_{27} + w_{29} + w_{49} + w_{46}$

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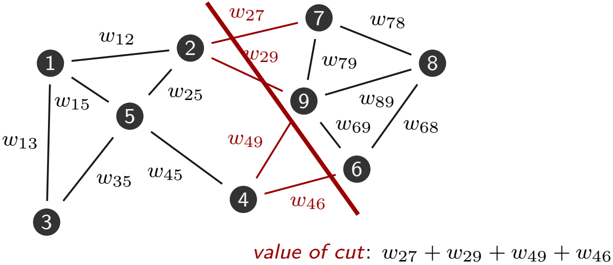

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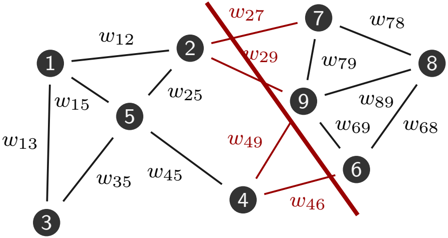

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Difficulty of optimization problems

- Closed-form solution ***(easy)***

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- Combinatorial, NP-Hard, requires exhaustive search ***(hard)***

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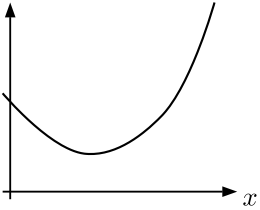
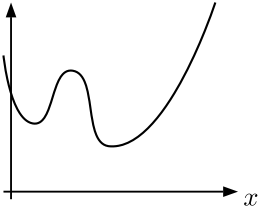
convex



nonconvex



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- Solved *efficiently* (polynomial-time algorithms)
- Lots of *applications:* machine learning, communications, economics and finance, control systems, electronic circuit design, statistics, etc.
- Many algorithms for *nonconvex optimization* use convex surrogates

Convex problems

Hierarchical classification (specialized solvers):

Convex problems

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LP — linear programming

Convex problems

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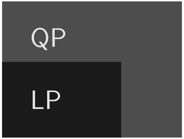

quadratic programming

linear programming

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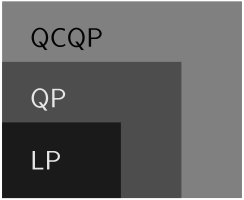

quadratically constrained QP

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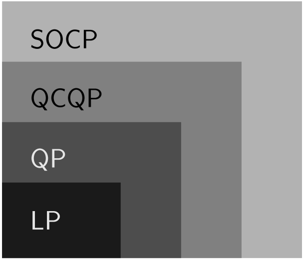

second-order cone programming

quadratically constrained QP

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linear programming

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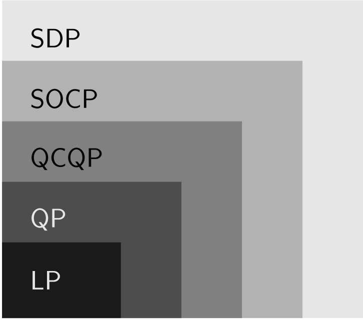

semidefinite programming

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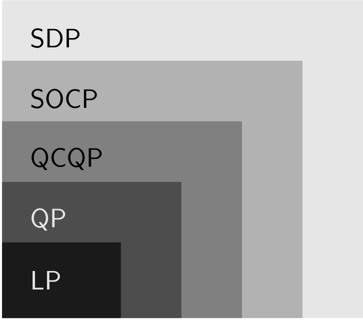

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Other classifications:

differentiable vs. *nondifferentiable* programming

unconstrained vs. *constrained* programming

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

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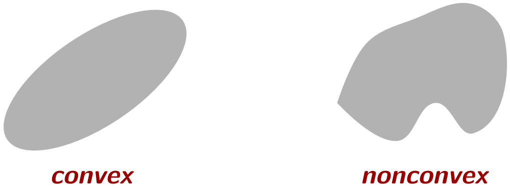
$x \in \Omega$ —— *convex set*

Convex sets

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \Omega \quad \text{——— convex set} \end{array}$$



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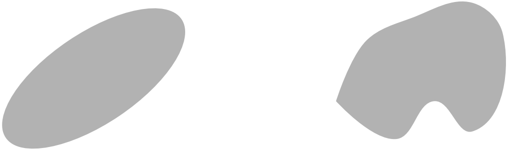

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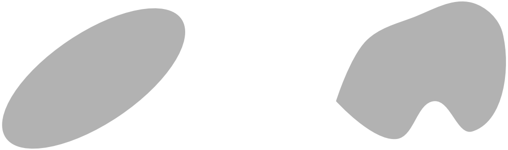

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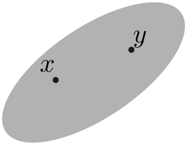

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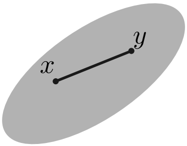

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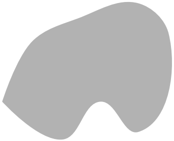

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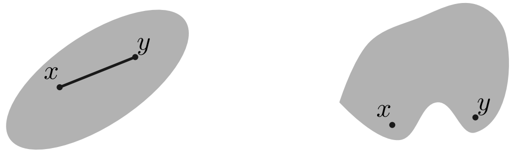

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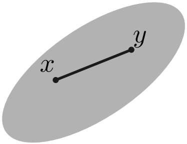

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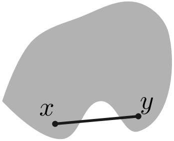

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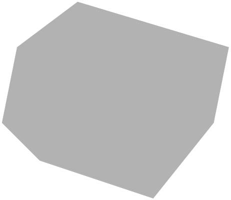

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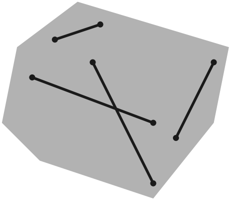

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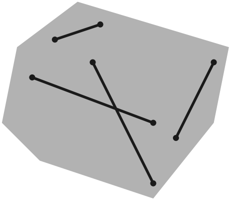

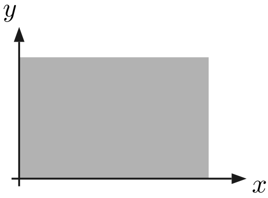

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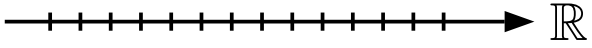

discrete sets

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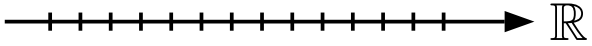

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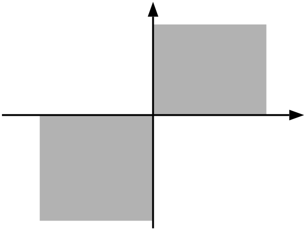

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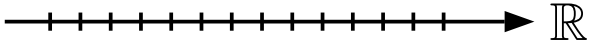

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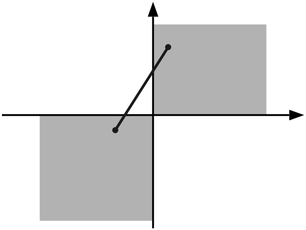

How to identify convex sets?

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vocabulary $+$ **grammar**

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simple sets

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simple sets — *operations preserving convexity*

Simple sets

Simple sets

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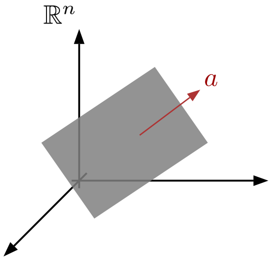

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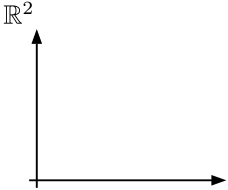

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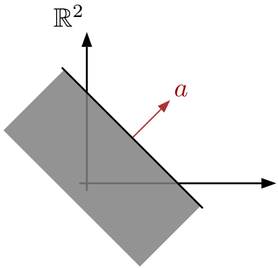

Simple sets

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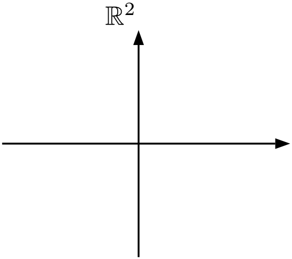

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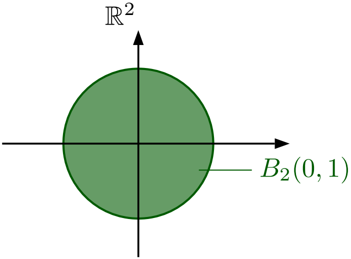

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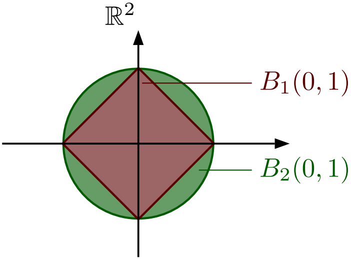

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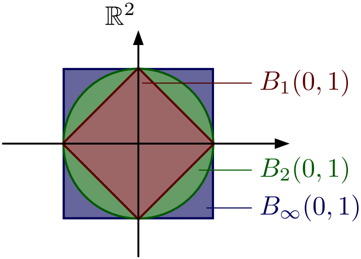

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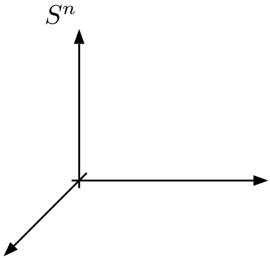

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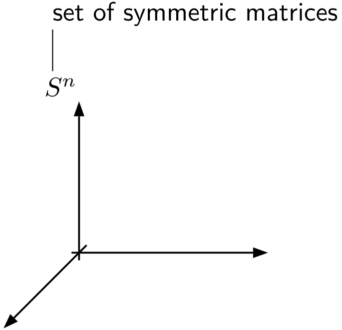

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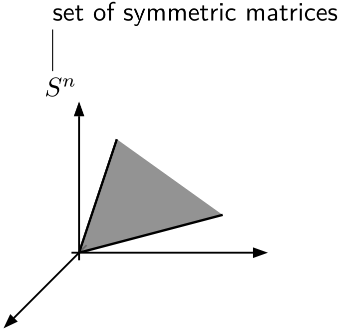

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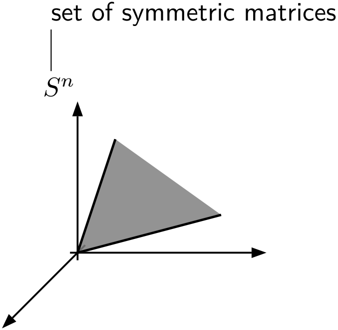

$$X \succeq 0_{n\times n} \quad \Longleftrightarrow \quad \lambda_{\min}(X) \geq 0 \quad \Longleftrightarrow \quad v^\top X v \geq 0\,, \quad \forall v$$

How to identify convex sets?

vocabulary $+$ **grammar**

simple sets — *operations preserving convexity*

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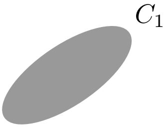

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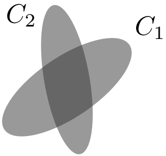

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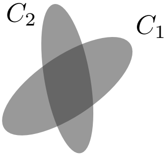

Intersection

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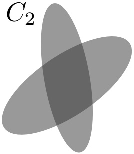

Intersection

$C_1, C_2, \ldots, C_m$: convex

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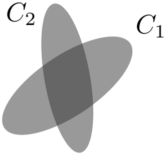

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$C_1, C_2, \ldots, C_m$: convex $\quad\Longrightarrow\quad$ $C_1 \cap C_2 \cap \cdots \cap C_m$: convex

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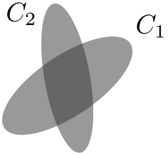

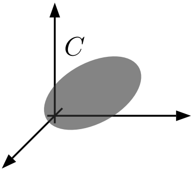

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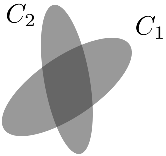

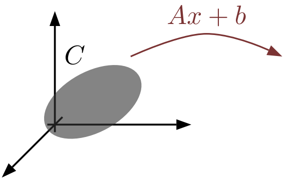

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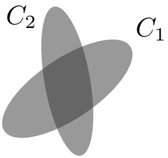

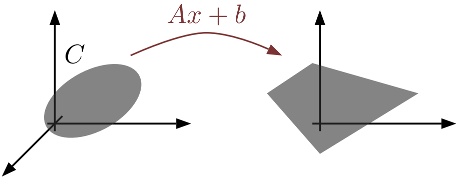

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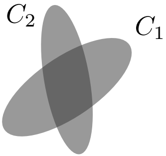

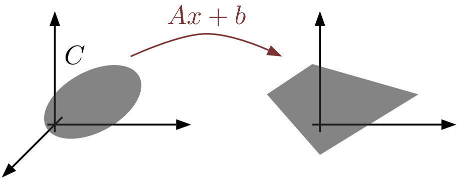

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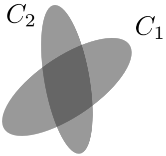
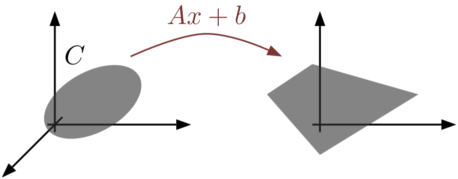
Affine operations

How to identify convex sets?





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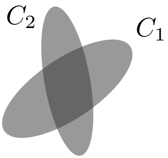

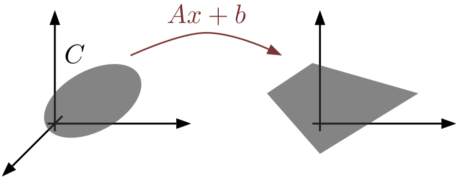

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Example

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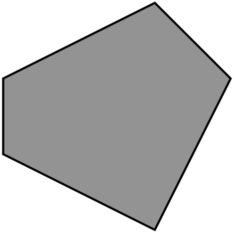

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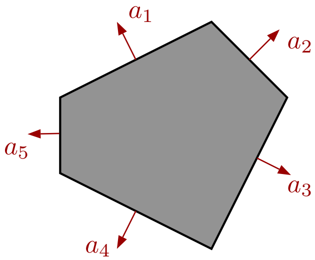

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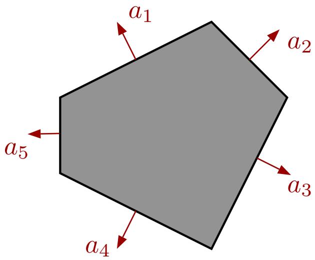
$$= \bigcap_{i=1}^{m} \mathcal{H}^-_{a_i, b_i}$$



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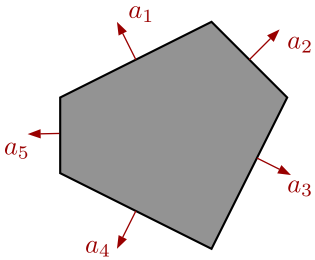

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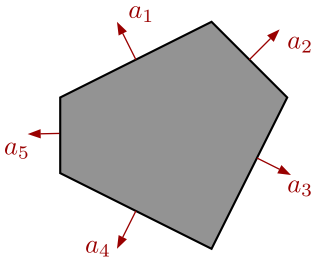

Example

Ellipsoids

Example

Ellipsoids $\big(A \succ 0\big)$

$$\mathcal{E} = \Big\{x \,:\, (x-c)^\top A^{-1}(x-c) \le 1\Big\}$$

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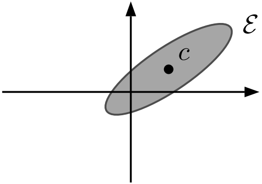

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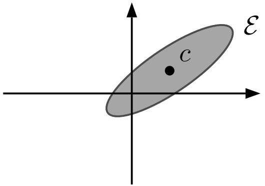

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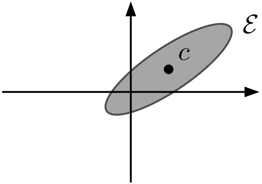

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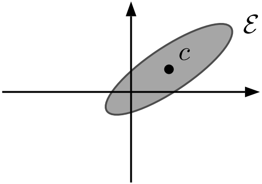

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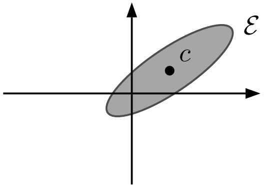

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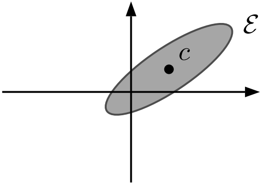

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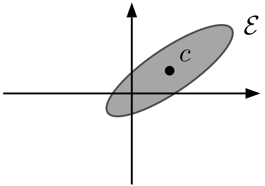

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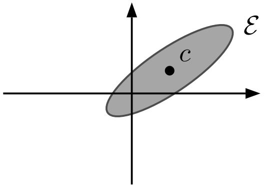

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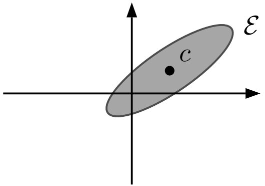

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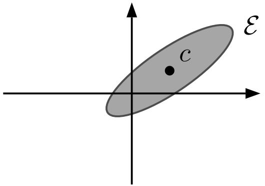

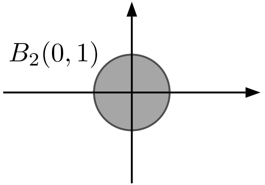

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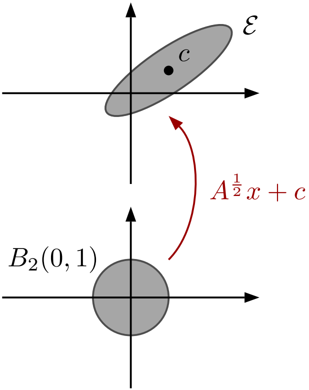

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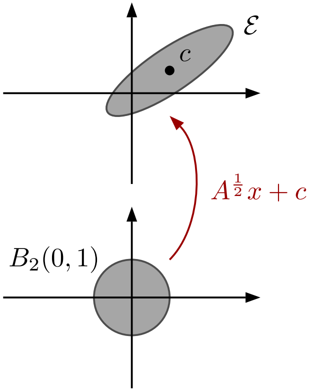

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Filter design constraints

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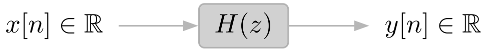
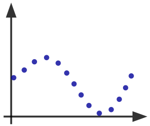


Example

Filter design constraints

$x[n] \in \mathbb{R}$ → $H(z)$ → $y[n] \in \mathbb{R}$



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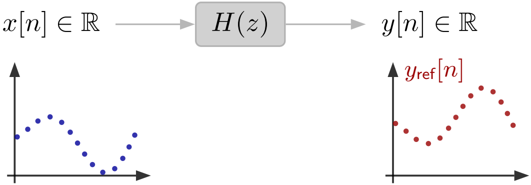

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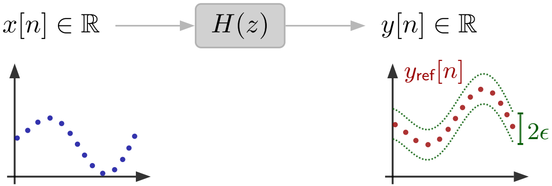

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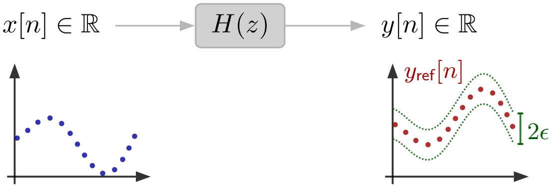

Goal: design $H(z)$ such that $\max_n \left|y[n] - y_{\text{ref}}[n]\right| \leq \epsilon$ for a fixed $x[n]$

Example

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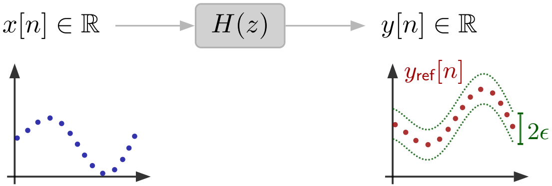

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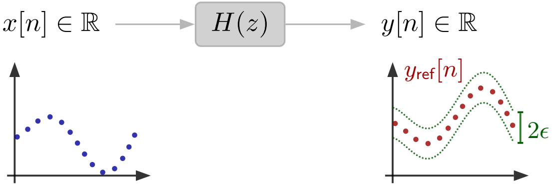
Assume finite impulse response (FIR):

Example

Filter design constraints



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Assume finite impulse response (FIR):

$$y[n] = h_0 \, x[n] + h_1 \, x[n-1] + \cdots + h_d \, x[n-d] \,, \quad n = 1, \ldots, N$$

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Matrix form:

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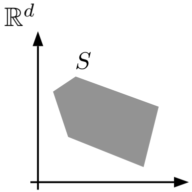

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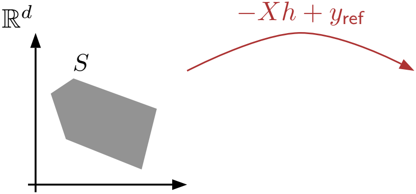

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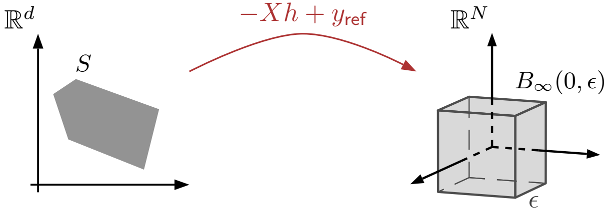

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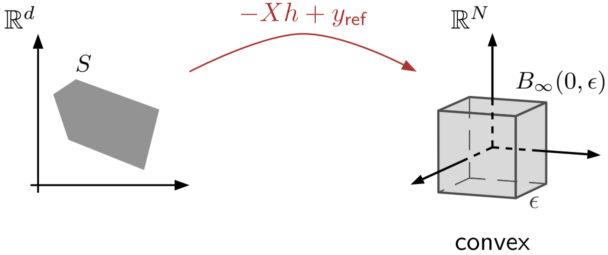

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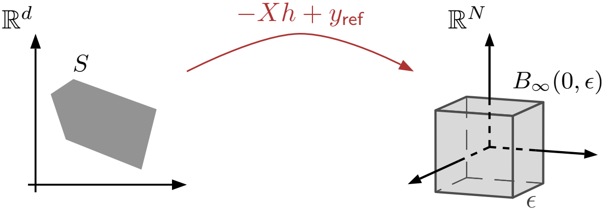

convex $\Longleftarrow$ convex

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

Convex functions

Convex functions

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

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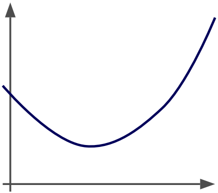

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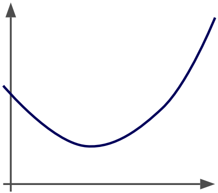

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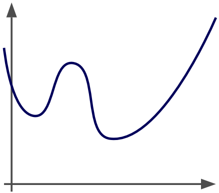
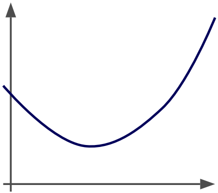


nonconvex

Convex functions

minimize$_x$ $f(x)$ —— *convex function*

subject to $x \in \Omega$



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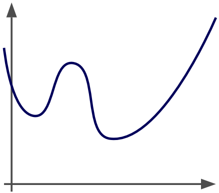


nonconvex

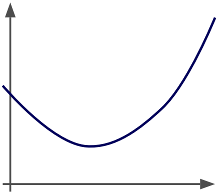
Definition:

f : dom $f \subseteq \mathbb{R}^n \to \mathbb{R}$ is *convex* when for any $x, y \in$ dom f,

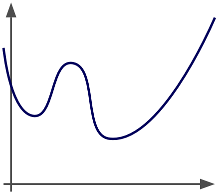
Convex functions

minimize$_x$ $f(x)$ —— *convex function*

subject to $x \in \Omega$



convex



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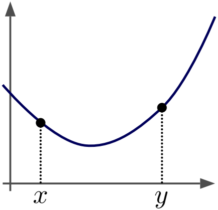

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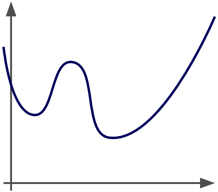

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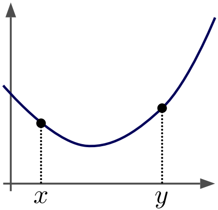

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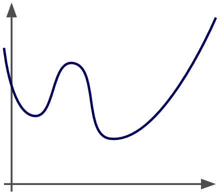

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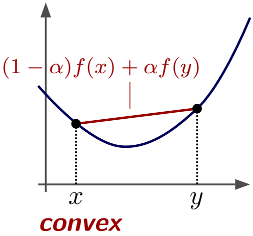

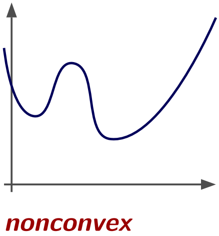

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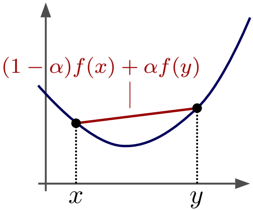

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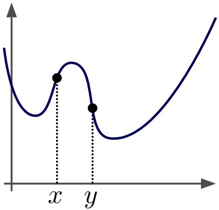

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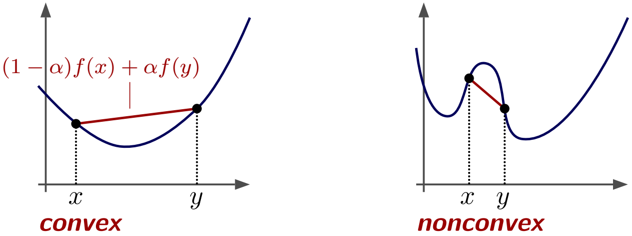

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How to identify convex functions?

vocabulary $+$ **grammar**

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vocabulary	+	**grammar**
definition		*operations preserving convexity*
differentiability conds.		
1D convexity		

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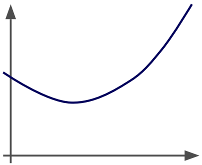

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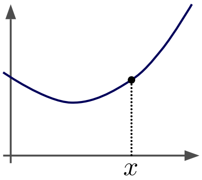

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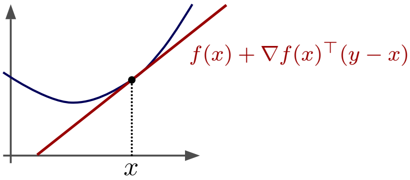

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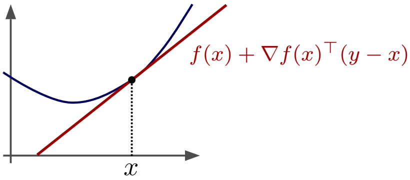

- When f is twice-differentiable,

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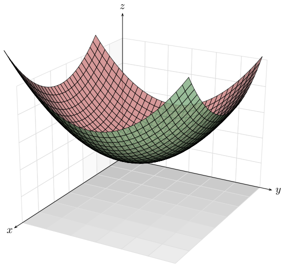

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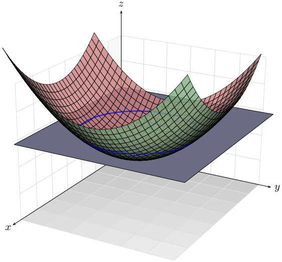

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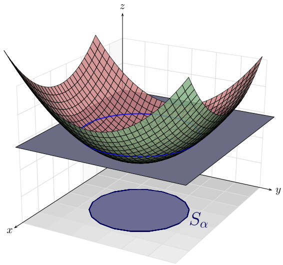

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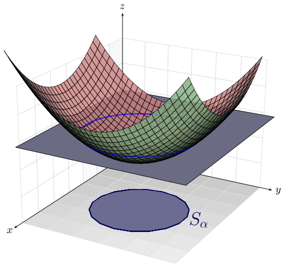

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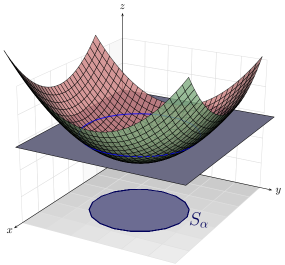

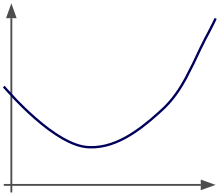

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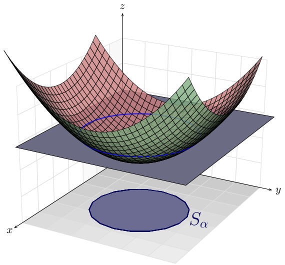

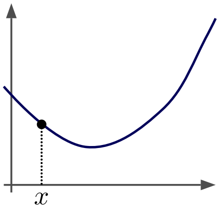

Equivalent definitions of convexity

$$f\big((1-\alpha)x + \alpha y\big) \le (1-\alpha) f(x) + \alpha f(y)\,, \qquad \forall_{x,y \in \mathsf{dom}\, f}\,, \;\; \alpha \in [0,1]$$

Equivalent statements

- $g(t) = f(x + ty) \quad (g : \mathbb{R} \to \mathbb{R})$ is convex for all $x + ty \in \mathsf{dom}\, f$
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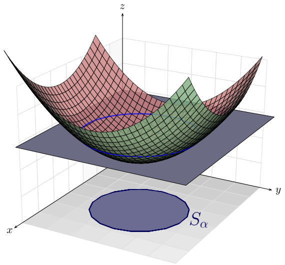

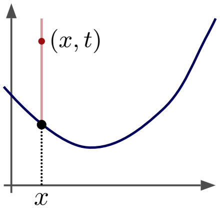

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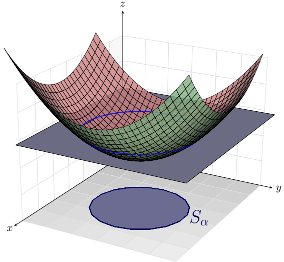

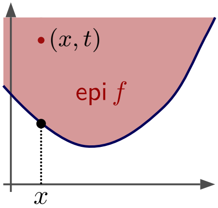

How to identify convex functions?

vocabulary + **grammar**

definition

differentiability conds.

1D convexity

operations preserving convexity

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Operations preserving convexity

Operations preserving convexity

Nonnegative weighted sums

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$$\begin{cases} f_1, \dots, f_m \ : \text{convex} \\ w_1, \dots, w_m \geq 0 \end{cases}$$

Operations preserving convexity

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Operations preserving convexity

Nonnegative weighted sums

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Precomposition with affine maps

Operations preserving convexity

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Operations preserving convexity

The set I is arbitrary: can even be uncountable

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Examples

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- $f(x) = \left\|y - Ax\right\|_2^2$

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- $f(x) = \|y - Ax\|_1$

Examples

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Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

Convex optimization problems

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$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0\,, \quad i = 1, \ldots, m \\ & h_i(x) = 0\,, \quad i = 1, \ldots, p \end{aligned}$$

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unbounded ($-\infty$), *infeasible* ($+\infty$)

A minimizer: $$x^\star \in \operatorname*{argmin}_x \; f(x) \quad \text{s.t.} \quad x \in \Omega$$

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Theorem

In convex problems, a *local* minimizer is always a *global* minimizer.

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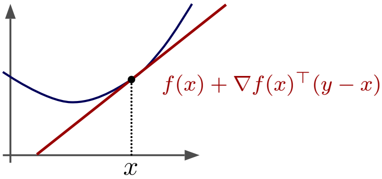

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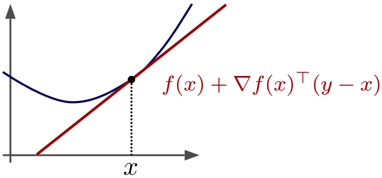

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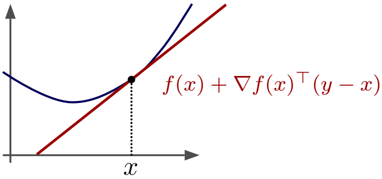

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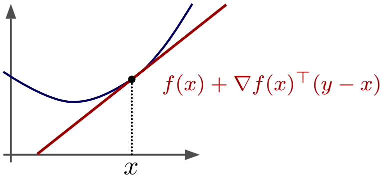

$\square$

Equivalence between optimization problems

Equivalence between optimization problems

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in X \end{array} \qquad \text{(P1)}$$

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Equivalence between optimization problems

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Equivalence between optimization problems

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(P1) and (P2) are *equivalent* when

- Given a solution $x^\star$ of (P1) we can obtain a solution $y^\star$ of (P2)
- Given a solution $y^\star$ of (P2) we can obtain a solution $x^\star$ of (P1)

Examples

Examples

$$\underset{x}{\text{maximize}} \quad f(x)$$

Examples

$$\underset{x}{\text{maximize}} \quad f(x) \qquad \Longleftrightarrow \qquad \underset{x}{\text{minimize}} \quad -f(x)$$

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$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x > 0 \end{array}$$

Examples

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$$\underset{x}{\text{minimize}} \quad a\, f(x) + b$$

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$a > 0$

$$\underset{x}{\text{minimize}} \quad f(x)$$

f convex

Examples

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$$\underset{x}{\text{minimize}} \quad \underbrace{f(x)}_{\text{convex}} \qquad \Longleftrightarrow \qquad \underset{x}{\text{minimize}} \quad g\big(f(x)\big)$$

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($a > 0$)

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($f(x)$: convex)

if $\text{im}\, f \subseteq \text{dom}\, g$

and $g \circ f$: convex

Air Traffic Control

Air Traffic Control

- n airplanes land in order $1, 2, \ldots, n$
- t_i: arrival time of airplane i
- Airplane i has to land in interval $[m_i, M_i]$

Air Traffic Control

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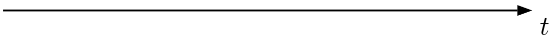

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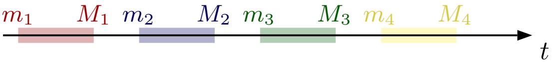

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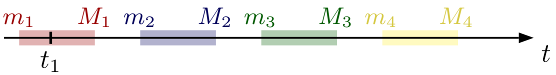

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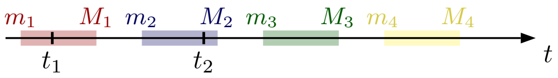

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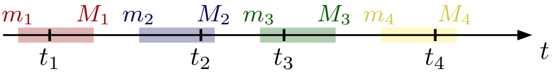

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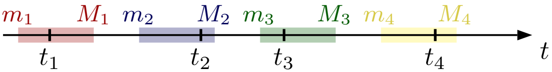

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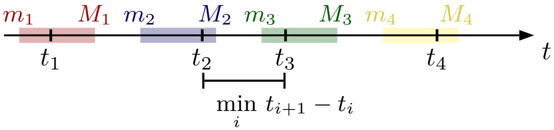

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Convex

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$$A = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 1 & -1 & \cdots & 0 & 0 & -1 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & -1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -m_1 \\ M_1 \\ \vdots \\ -m_n \\ M_n \end{bmatrix}$$

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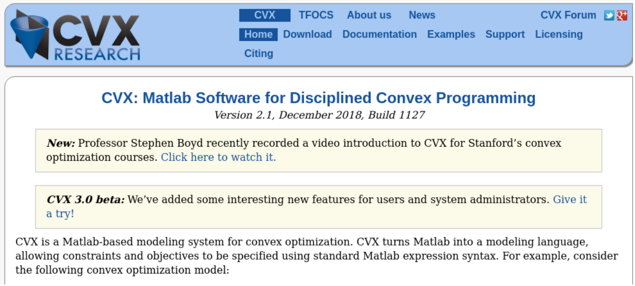
CVX (cvxr.com/cvx) manipulates and solves *convex* problems



Air Traffic Control

```
cvx_begin
    variables t1 t2 t3 t4 t5;
    maximize( min([t2-t1, t3-t2, t4-t3, t5-t4]) );
    subject to
        1 <= t1 <= 2;
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$$(t_1^\star\,,\ t_2^\star\,,\ t_3^\star\,,\ t_4^\star\,,\ t_5^\star) = (1\,,\ 3.25\,,\ 5.5\,,\ 7.75\,,\ 10)$$

Portfolio Optimization

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Convex Quadratic Program (QP)

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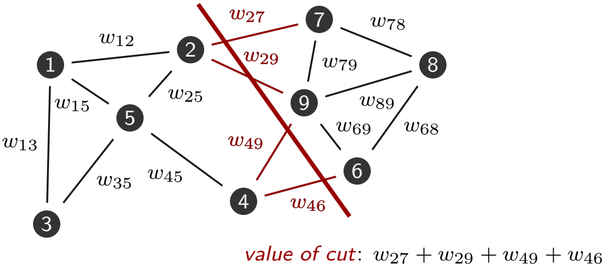
Convex QP

MAXCUT



Cut: set of edges whose removal splits the graph into two

MAXCUT problem: find the cut with maximum weight

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{maximize}} & \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{1 - x_i x_j}{2} \\ \text{subject to} & x_i \in \{-1, 1\}, \qquad i = 1, \ldots, n. \end{array}$$

$$
\begin{aligned}
p^\star = \quad & \max_{x} \quad \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\,\frac{1-x_i x_j}{2} \\
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\begin{aligned}
= -\ \min_x \quad & \frac{1}{4}\left(x^\top W x - 1_n^\top W 1_n\right) \\
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$$

$$
\begin{aligned}
p^\star = \ & \max_x \quad \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\,\frac{1 - x_i x_j}{2} \\
& \text{s.t.} \quad x_i \in \{-1, 1\}\,, \qquad i = 1, \ldots, n
\end{aligned}
$$

$$
\begin{aligned}
= - & \min_x \quad \frac{1}{4}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} x_i x_j - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\right] \\
& \text{s.t.} \quad x_i^2 = 1\,, \qquad i = 1, \ldots, n
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$$
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= - & \min_x \quad \frac{1}{4}\left(x^\top W x - 1_n^\top W 1_n\right) \\
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$W \in \mathbb{R}^{n \times n}$: weighted adjacency matrix

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\begin{aligned}
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Relax . . .

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Convex Semi-Definite Program (SDP)

Procedure: (Goemans & Williamson, 95')

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It can be shown that

$$d^\star \,\geq\, p^\star \,\geq\, \mathbb{E}[C] \,\geq\, 0.87856\, d^\star$$

Solving Optimization Problems

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Large-scale problems & real-time solutions require tailored solvers

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For $n = 5000$ and $m = 500$,

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For $n = 5000$ and $m = 500$,

- CVX: 56.16 s
- SPGL1: 0.82 s (tailored solver)

Optimization Algorithms

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- **Unconstrained & differentiable**

  Gradient descent (and Nesterov's acceleration scheme)

  Block coordinate descent

  Newton (and approximations)

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And now, *neural networks!*

Nesterov's acceleration scheme

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- $f^\star := \min_x \; f(x) > -\infty$

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Example

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \log \sum_{i=1}^{m} \exp\left(a_i^\top x + b\right)$$

randomly generated data with $m = 500$, $n = 50$

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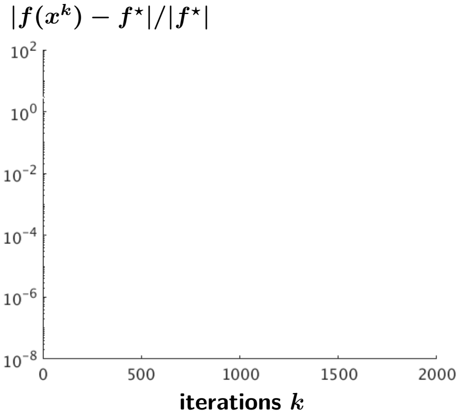

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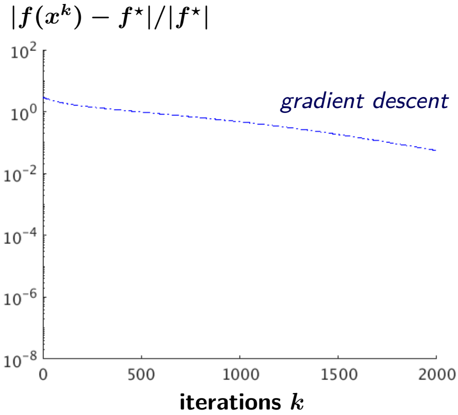

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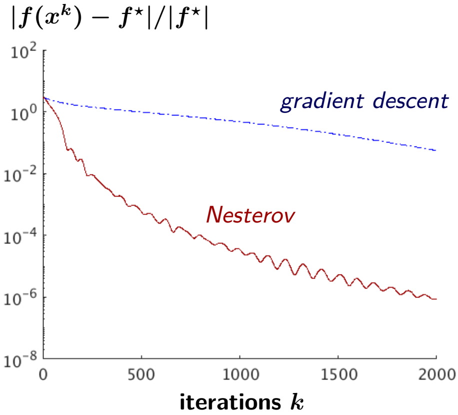

Outline

Convex sets

Identifying convex sets

Examples: geometrical sets and filter design constraints

Convex functions

Identifying convex functions

Relation to convex sets

Optimization problems

Convex problems, properties, and problem manipulation

Examples and solvers

Statistical estimation

Maximum likelihood & maximum a posteriori

Nonparametric estimation

Hypothesis testing & optimal detection

Maximum likelihood

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negative log-likelihood

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The a_i's will denote the rows of $A = \begin{bmatrix} — & a_1^\top & — \\ & \vdots & \\ — & a_m^\top & — \end{bmatrix} \in \mathbb{R}^{m \times n}$

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Common noise densities:

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Gaussian: $V \sim \mathcal{N}(0, \sigma^2)$ $\qquad f_V(v) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$

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Convex

Maximum likelihood

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Laplacian:

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feasibility problem (convex)

Example with a discrete RV

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$Y \in \mathbb{N}$: # of traffic accidents in a given period

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Goal:

Given m independent observations $\left\{\left(U^{(i)}, Y^{(i)}\right)\right\}_{i=1}^{m}$, estimate a and b.

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Joint probability mass function: $p_{YU}(y, u\,;\, a, b) = \frac{e^{-(a^\top u+b)}\,(a^\top u+b)^y}{y\,!}$

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Convex

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- Observations: $Y_i = a_i^\top X + V_i, \;\; i = 1, \ldots, m \;,\; \&\; V_i \overset{\text{iid}}{=} V \sim \mathcal{U}(-c, \, c)$

$$f_{Y_1 \cdots Y_m | X}(y_1, \ldots, y_m | x) \;=\; \prod_{i=1}^{m} f_{Y_i|X}(y_i|x) \;=\; \prod_{i=1}^{m} \mathcal{U}\big(a_i^\top x - c \,,\, a_i^\top x + c\big)$$

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Convex

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ML and MAP allow estimating parameters of given distributions

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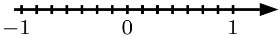

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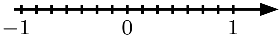

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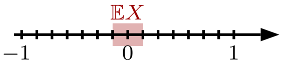

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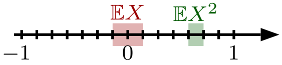

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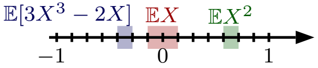

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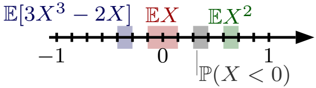

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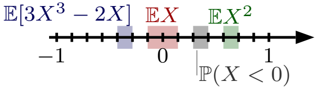

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Find a distribution satisfying these constraints & with maximum entropy

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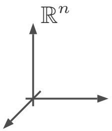

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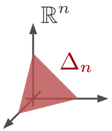

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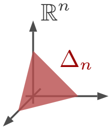

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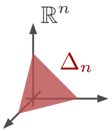

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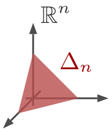

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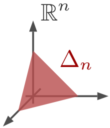

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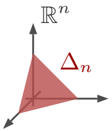

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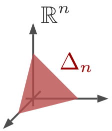

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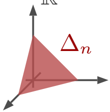

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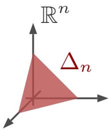

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All constraints are *linear inequalities* in p!

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Optimization problem: *(convex)*

$$\begin{aligned}
\underset{p \in \mathbb{R}^{100}}{\text{minimize}} \quad & \textstyle\sum_{i=1}^{n} p_i \log p_i \\
\text{subject to} \quad & -0.1 \le \alpha^\top p \le 0.1 \\
& 0.5 \le \beta^\top p \le 0.6 \\
& -0.3 \le \gamma^\top p \le -0.2 \\
& 0.3 \le \sigma^\top p \le 0.4
\end{aligned}$$

Nonparametric estimation

```
n = 100;
alpha = linspace(-1,1,n)';
cvx_begin
    variable p(n,1);
    minimize( -sum( entr( p ) ) );
    subject to
        p >= 0;
        ones(1, n)*p == 1;
        -0.1 <= alpha'*p <= 0.1;
        0.5 <= (alpha.^2)'*p <= 0.6;
        -0.3 <= (3*alpha.^3 - 2*alpha)'*p <= -0.2;
        0.3 <= (alpha < 0)'*p <= 0.4;
cvx_end
```

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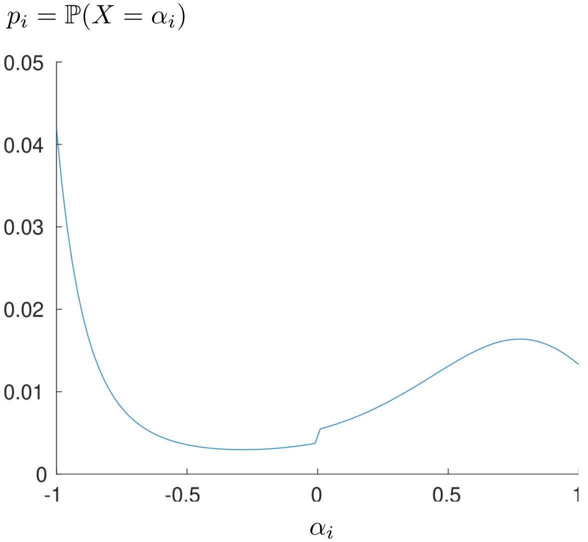

Hypothesis testing & optimal detection

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X : discrete random variable w/ probability mass function (pmf) p_θ

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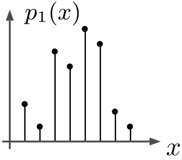

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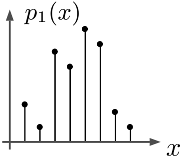

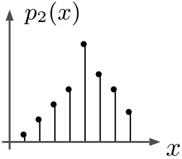

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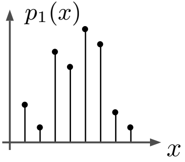

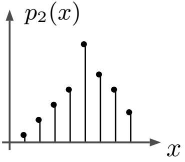

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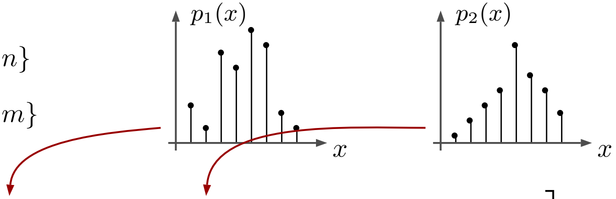

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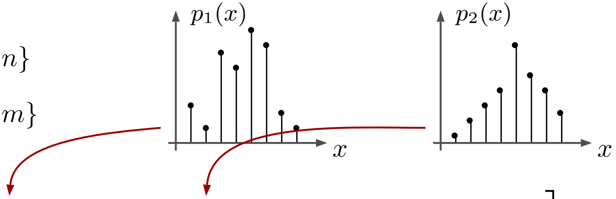

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Goal: Estimate θ based on an observation of X

Detector:

Detector: $\Psi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$

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If each t_i is a canonical vector $(0,\ldots,1,\ldots,0)$, then T is deterministic

Detection probability matrix:

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Multi-objective optimization

Minimax detector

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Convex

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  where c_i: ith column of $C := W P^\top$

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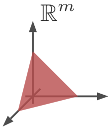

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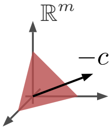

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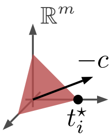

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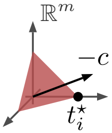

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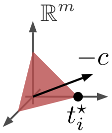

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Therefore,

$$c_i^\top t_i = t_{1i}\,W_{12}\,\mathbb{P}(X=i\,|\,\theta=2) + t_{2i}\,W_{21}\,\mathbb{P}(X=i\,|\,\theta=1)$$

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Neyman-Pearson lemma:

For each $\alpha > 0$, the likelihood-ratio test yields a (deterministic) Pareto-optimal detector.

Deterministic vs randomized detectors

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Minimax detector (random):

$$T_{\mathsf{MM}} = \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 0 & 1/3 & 1 & 1 \end{bmatrix}$$

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Receiver operating characteristic (ROC)

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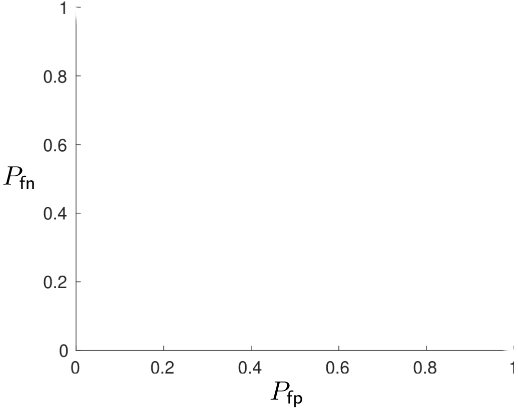

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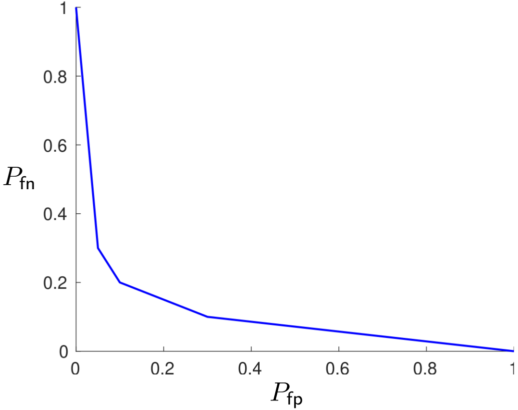

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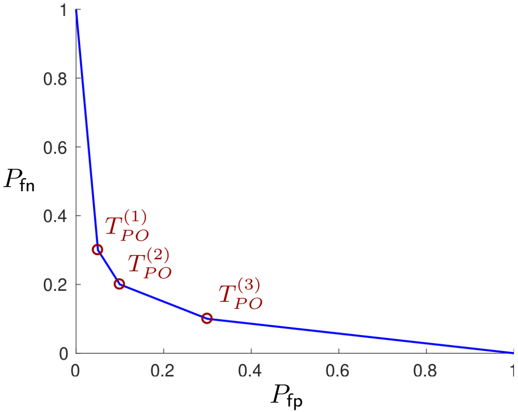

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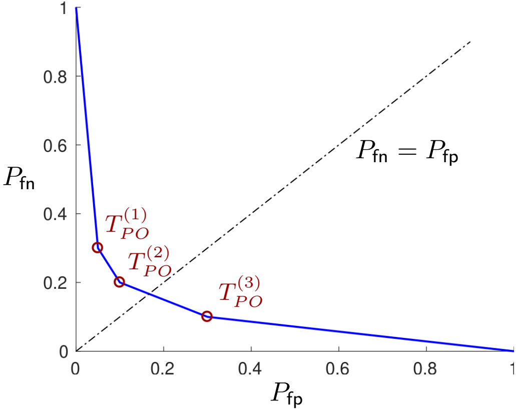

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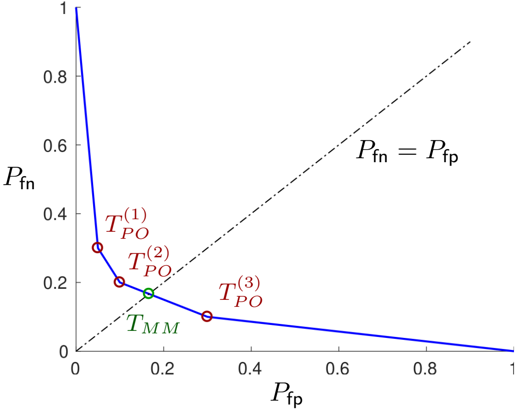

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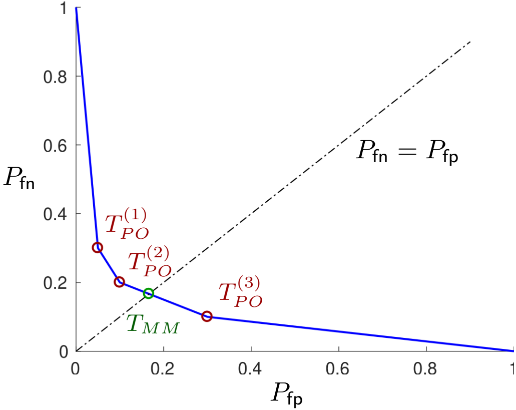

Minimax estimator has $(P_{\mathsf{fp}}, P_{\mathsf{fn}}) = (\frac{1}{6}, \frac{1}{6})$ and outperforms any deterministic estimator

Conclusions

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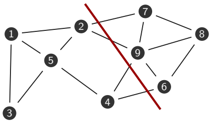

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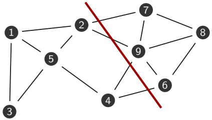

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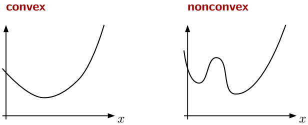

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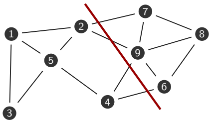

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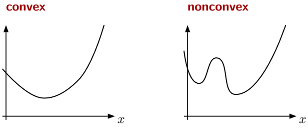

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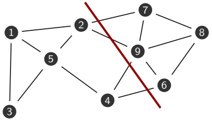

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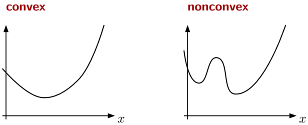

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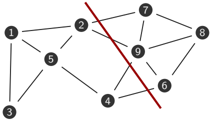

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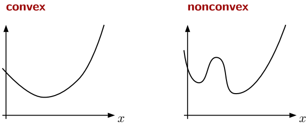

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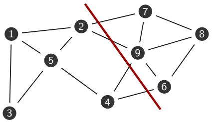

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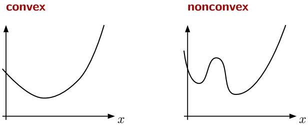
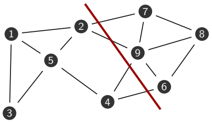


- Statistical estimation:

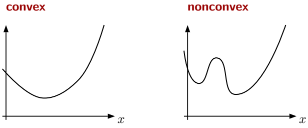
  Find parameters of distributions (ML/MAP)

  Entire distributions (nonparametric)

Conclusions



- Optimization problems arise in many areas
- Essential to distinguish easy *(convex)* from hard *(nonconvex)* probs
- Use CVX/CVXPY/Convex for small-scale problems
- *Didn't cover* optimality conditions & theory
- *Didn't cover* optimization algorithms



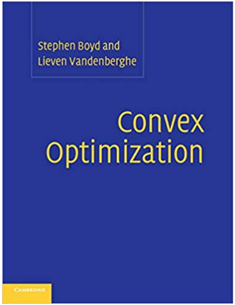
- Statistical estimation:

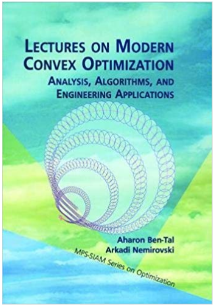
  Find parameters of distributions (ML/MAP)

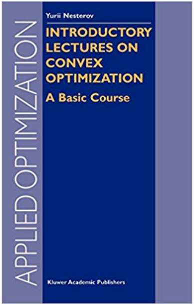
  Entire distributions (nonparametric)

- Multiple hypothesis testing via optimization

References and Resources







Lectures:

- web.stanford.edu/∼boyd/cvxbook/
- users.isr.ist.utl.pt/∼jxavier/NonlinearOptimization18799-2018
- www.seas.ucla.edu/∼vandenbe/ee236c